ON THE CONCEPT OF CONVERGENCE OF CONTINUED FRACTIONS

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A continued fraction is an expression of the form

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}},$$

where a_n, b_n are real or complex numbers and $a_n \neq 0$. A standard definition (see e.g., Jones and Thron [1]) says that a continued fraction converges, if the sequence of its truncated continued fractions converges:

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots = \lim_{n\to\infty} b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots + \frac{a_n}{b_n}$$

We argue that there is a less restrictive and more natural concept of convergence of continued fractions. A continued fraction represents an infinite product of Möbius transformations. A real regular Möbius transformation $M : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ on the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is defined by $M(x) = \frac{ax+b}{cx+d}$, $M(\infty) = \frac{a}{c}$, $M(-\frac{d}{c}) = \infty$. Here $a, b, c, d \in \mathbb{R}$ and $\det(M) = ad - bc \neq 0$. The transformation is determined by a (2×2)-matrix $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and the composition of transformations corresponds to the product of matrices. For a continued fraction $b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots$ we have transformations $M_0(x) = x + b_0$, $M_n(x) = \frac{a_n}{x+b_n}$ with matrices

$$M_0 = \begin{bmatrix} 1 & b_0 \\ 0 & 1 \end{bmatrix},\ M_n = \begin{bmatrix} 0 & a_n \\ 1 & b_n \end{bmatrix},\ n > 0$$

The convergents p_n, q_n are defined by $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = b_0$, $q_0 = 1$, $p_1 = a_1 + b_0 b_1$, $q_1 = b_1, \ldots,$ $p_n = a_n p_{n-2} + b_n p_{n-1}$, $q_n = a_n q_{n-2} + b_n q_{n-1}$. For the product matrices $R_n = M_0 \cdots M_n$ we get

$$\begin{aligned} R_0 &= \begin{bmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{bmatrix} = \begin{bmatrix} 1 & b_0 \\ 0 & 1 \end{bmatrix} = M_0, \\ R_n &= \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix} = \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & a_n \\ 1 & b_n \end{bmatrix} = R_{n-1} \cdot M_n \end{aligned}$$

Note that $R_n(0) = \frac{p_n}{q_n}$, $R_n(\infty) = \frac{p_{n-1}}{q_{n-1}}$, so the convergence of a continued fraction to $x \in \overline{\mathbb{R}}$ means that $\lim_{n\to\infty} R_n(z) = x$ for two particular values $z = 0, z = \infty$. However, in the context of transformations, the choice of these two particular values seems to be rather arbitrary. Consider a periodic continued fraction

$$\frac{2}{1+} \frac{1}{0+} \frac{2}{1+} \frac{1}{0+} \frac{2}{1+} \cdots.$$

with $M_0 = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$, $M_{2n+1} = \left[\begin{smallmatrix}0 & 2\\ 1 & 1\end{smallmatrix}\right]$, $M_{2n+2} = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$. For $n > 0$ we get

$$R_{2n-1} = \begin{bmatrix} 0 & 2^n \\ 1 & 2^n - 1 \end{bmatrix},\ R_{2n} = \begin{bmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{bmatrix}.$$

Thus $\frac{p_{2n-1}}{q_{2n-1}} = \frac{2^n}{2^n-1} \to 1$, $\frac{p_{2n}}{q_{2n}} = 0$, so the continued fraction does not converge according to the standard definition. However, we have

$$\begin{aligned} z \neq \infty \quad &\Rightarrow \quad \lim_{n\to\infty} R_{2n-1}(z) = \lim_{n\to\infty} \frac{2^n}{z + 2^n - 1} = 1 \\ z \neq 0 \quad &\Rightarrow \quad \lim_{n\to\infty} R_{2n}(z) = \lim_{n\to\infty} \frac{2^n z}{(2^n - 1)z + 1} = 1 \end{aligned}$$

Thus $\lim_{n\to\infty} R_n(z) = 1$ for all $z \in \overline{\mathbb{R}} \setminus \{0, \infty\}$. The real Möbius transformations act also on the extended complex plane $\overline{\mathbb{C}}$, and in our example we get $\lim_{n\to\infty} R_n(z) = 1$ for every $z \in \mathbb{C}$ with $\Re(z) \neq 0$. A similar thing happens with a continued fraction

$$\frac{2}{1+}\,\frac{1}{a_1}+\frac{2}{1+}\,\frac{1}{a_2}+\frac{2}{1+}\,\frac{1}{a_3}+\cdots$$

provided a_n converge sufficiently fast to 0. For example for $a_n = 4^{-n}$, the continued fraction is not convergent according to the standard definition, but $\lim_{n\to\infty} R_n(z) = 1.124\ldots$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ and for most of $z \in \mathbb{R}$. Thus a more natural and less restrictive concept of convergence of continued fractions is given by one of the equivalent conditions of Theorem 1:

Theorem 1. *Let $(R_n)_{n\geq 0}$ be a sequence of regular real Möbius transformations and $x \in \overline{\mathbb{R}}$. Then the following three conditions are equivalent.*
1. $\exists z \in \mathbb{C} \setminus \mathbb{R}, \lim_{n\to\infty} R_n(z) = x$.
2. $\forall z \in \mathbb{C} \setminus \mathbb{R}, \lim_{n\to\infty} R_n(z) = x$.
3. $\lim_{n\to\infty} R_n\mu = \delta_x$ for every Borel measure μ on $\overline{\mathbb{R}}$ which is absolutely continuous with respect to the Lebesgue measure.

Here δ_x is the Dirac point measure concentrated on x. The standard definition implies the condition of Theorem 1. This is shown in Theorem 2.

Theorem 2. *Assume that $(R_n)_{n\geq 0}$ is a sequence of real regular transformations, $v, w \in \overline{\mathbb{R}}$, $v \neq w$ and $\lim_{n\to\infty} M_n(v) = \lim_{n\to\infty} M_n(w) = x \in \overline{\mathbb{R}}$. Then $\forall z \in \mathbb{C} \setminus \mathbb{R}, \lim_{n\to\infty} M_n(z) = x$.*

References

[1] W. B. Jones and W. J. Thron. *Continued fractions*, volume 11 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 1984.

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