Piecewise isometry and substitutive language

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Abstract

In this talk we will present a class of piecewise isometries, where the symbolic dynamics can be described in terms of substitutions. This result is a generalisation of the results of [6].

Introduction We study an example of a piecewise isometry introduced in [3] by Boshernitzan and Goetz, this map is called a piecewise rotation. Actually it is the piecewise isometry which has been the most studied, see [6] and [5]. Consider a line in the plane, two points outside the line and an angle. The map is locally defined on each half plane (defined by the line) by a rotation of the fixed angle centered in one of the two points. The phase space of the map can be described with two real parameters: one for the angle and one which measure the relative position of the centers. If the centers are centrally symmetric through the origin, then this parameter is null and the map is called symmetric. Thus the map can always been written in the following form

$$T : \mathbb{C} \to \mathbb{C}$$

$$T(z) = \begin{cases} e^{2i\pi\theta}(z+\sigma+1) & Imz > 0 \\ e^{2i\pi\theta}(z+\sigma-1) & Imz < 0 \end{cases}$$

where σ is a real number. The parameter θ is called the angle of the map by abuse of notation.

Among the positions of the centers of rotations this map can be bijective, non injective, non surjective. In [3] Boshernitzan and Goetz show that in the two last cases the map is either globally attractive or globally repulsive. In the bijective case, Goetz and Quas showed that for a rational angle θ every orbit is bounded, see [6]. In order to prove this result they introduce a symbolic study of this map associated to the coding according to the two half-planes. They define the notion of rotationally coded points, which represent points which have the same symbolic coding as points under the action of a rotation of the fixed angle, centered at the origin. This notion is very useful to find periodic orbits, but it only works for maps closed to the symmetric ones. In the irrational case Goetz, Quas and Cheung are able to give precise bounds on the density of periodic orbits, see [5]. Remark that a similar map has also been studied in [1].

Results of the paper: Here we want to give a precise description of the symbolic dynamics. We do not want to restrict to bijective cases or to symmetric cases inside the bijective case. Nevertheless we restrict our study to a finite family of angles

$$\{\frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \frac{1}{8}\}.$$

For the cases $\frac{2\pi}{5}, \frac{\pi}{4}$ we find some bounded orbits which are not periodic. Thus these orbits does not come from rotationally coded orbits.

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Words, substitutions: Our description of the symbolic dynamics is made in terms of substitutions. A **substitution** σ is an application from an alphabet $\mathcal{A}$ to the set $\mathcal{A}^* \setminus \{\varepsilon\}$ of nonempty finite words on $\mathcal{A}$. It extends to a morphism of $\mathcal{A}^*$ by concatenation, that is $\sigma(uv) = \sigma(u)\sigma(v)$. Let S be a finite set of substitutions and X a finite set of words. A **substitutive language** (or S adic system) is a language where every word is a factor of a word obtained by composition of several substitutions of S applied to a word of X. This class of words has been studied in combinatorics of words, see for example [7] and [2]. Here we give an example of a geometric dynamical system which has a symbolic description in term of a substitutive language.

Theorem 1 *The symbolic dynamics of a piecewise rotation has the following properties:*

- *If* $\theta \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{6}\}$, *every orbit is periodic.*
- *If* $\theta \in \{\frac{1}{5}, \frac{1}{8}\}$, *every orbit is bounded but some are non periodic.*
- *For non symmetric cases, the dynamics is the same for all values of a rational number* $\sigma \in (0, 1)$.

In all the cases the language is substitutive.

In the figures we show some pictures of the dynamic of this map for different values of the parameters.

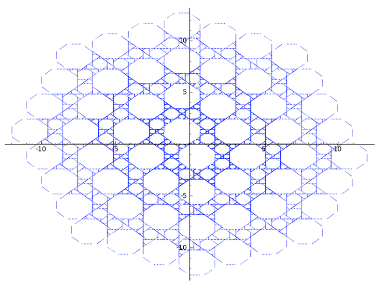

Figure 1: Cellular decomposition for the symmetric map of angle $\frac{1}{8}$.

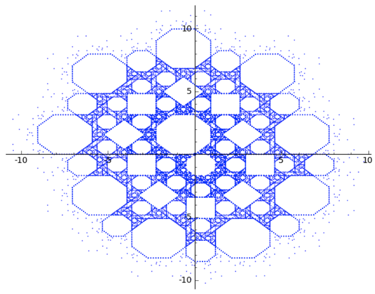

Figure 2: Non symmetric case for the angle $\frac{\pi}{4}$.

This is a joint work with Idrissa Kabore, see [4].

References

[1] S. Akiyama and A. Pethő. Discretized rotation has infinitely many periodic orbits. *Nonlinearity*, 26(3):871–880, 2013.

[2] V. Berthé and V. Delecroix. S-adic expansions: a combinatorial, arithmetic and geometric approach. *RIMS Lecture note 'Kokyuroku Bessatu'*, To appear.

[3] M. Boshernitzan and A. Goetz. A dichotomy for a two-parameter piecewise rotation. *Ergodic Theory Dynam. Systems*, 23(3):759–770, 2003.

[4] N. Bédaride and I. Kaboré. Piecewise isometries: symbolic dynamics. *Arxiv: 1306.6150*, To appear.

[5] Y. Cheung, A. Goetz, and A. Quas. Piecewise isometries, uniform distribution and $3\log 2 - \pi^2/8$. *Ergodic Theory Dynam. Systems*, 32(6):1862–1888, 2012.

[6] A. Goetz and A. Quas. Global properties of a family of piecewise isometries. *Ergodic Theory Dynam. Systems*, 29(2):545–568, 2009.

[7] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.