A new approach to the 2-regularity of the ℓ-abelian complexity of 2-automatic sequences (extended abstract)

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Abstract

We show that a sequence satisfying a certain symmetry property is 2-regular in the sense of Allouche and Shallit. We apply this theorem to develop a general approach for studying the ℓ-abelian complexity of 2-automatic sequences. In particular, we prove that the period-doubling word and the Thue–Morse word have 2-abelian complexity sequences that are 2-regular. Along the way, we also prove that the 2-block codings of these two words have 1-abelian complexity sequences that are 2-regular.

1 Introduction

This extended abstract[1] is about some structural properties of integer sequences that occur naturally in combinatorics on words. Since the fundamental work of Cobham [6], the so-called automatic sequences have been extensively studied. We refer the reader to [3] for basic definitions and properties. These infinite words over a finite alphabet can be obtained by iterating a prolongable morphism of constant length to get an infinite word (and then, an extra letter-to-letter morphism, also called coding, may be applied once). As a fundamental example, the *Thue–Morse word* $\mathbf{t} = \sigma^\omega(0) = 0110100110010110\cdots$ is a fixed point of the morphism σ over the free monoid $\{0,1\}^*$ defined by $\sigma(0) = 01$, $\sigma(1) = 10$. Similarly, the *period-doubling word* $\mathbf{p} = \psi^\omega(0) = 0100010101000100010\cdots$ is a fixed point of the morphism ψ over $\{0,1\}^*$ defined by $\psi(0) = 01$, $\psi(1) = 00$.

Let $k \geq 2$ be an integer. One characterization of k-automatic sequences is that their k-kernels are finite; see [7] or [3, Section 6.6].

Definition 1. The *k-kernel* of a sequence $\mathbf{s} = s(n)_{n\geq 0}$ is the set

$$\mathcal{K}_k(\mathbf{s}) = \{s(k^i n + j)_{n\geq 0} : i \geq 0 \text{ and } 0 \leq j < k^i\}.$$

For instance, the 2-kernel $\mathcal{K}_2(\mathbf{t})$ of the Thue–Morse word contains exactly two elements, namely $\mathbf{t}$ and $\sigma^\omega(1)$.

A natural generalization of automatic sequences to sequences on an infinite alphabet is given by the notion of k-regular sequences. We will restrict ourselves to sequences taking integer values only.

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[1]For the full version of this paper, see [15].

Definition 2. Let $k \geq 2$ be an integer. A sequence $\mathbf{s} = s(n)_{n\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is *k-regular* if $\langle \mathcal{K}_k(\mathbf{s}) \rangle$ is a finitely-generated $\mathbb{Z}$-module, i.e., there exist a finite number of sequences $s_1(n)_{n\geq 0}, \ldots, s_\ell(n)_{n\geq 0}$ such that every sequence in the k-kernel $\mathcal{K}_k(\mathbf{s})$ is a $\mathbb{Z}$-linear combination of the s_r's. Otherwise stated, for all $i \geq 0$ and for all $j \in \{0, \ldots, k^i - 1\}$, there exist integers $c_1, \ldots, c_\ell$ such that

$$\forall n \geq 0, \quad s(k^i n + j) = \sum_{r=1}^{\ell} c_r \, t_r(n).$$

Allouche and Shallit give many natural examples of k-regular sequences and classical results [1, 2]. The k-regularity of a sequence provides us with structural information about how the different terms are related to each other.

We will often make use of the following composition theorem for a function F defined piecewise on several k-automatic sets.

Lemma 3. *Let $k \geq 2$. Let $P_1, \ldots, P_\ell : \mathbb{N} \to \{0, 1\}$ be unary predicates that are k-automatic. Let $f_1, \ldots, f_\ell$ be k-regular functions. The function $F : \mathbb{N} \to \mathbb{N}$ defined by*

$$F(n) = \sum_{i=1}^{\ell} f_i(n) \, P_i(n)$$

is k-regular.

A classical measure of complexity of an infinite word $\mathbf{x}$ is its *factor complexity* $\mathcal{P}_{\mathbf{x}}^{(\infty)} : \mathbb{N} \to \mathbb{N}$ which maps n to the number of distinct factors of length n occurring in $\mathbf{x}$. It is well known that a k-automatic sequence $\mathbf{x}$ has a k-regular factor complexity function [13, 5]. As an example, again for the Thue–Morse word, we have

$$\mathcal{P}_{\mathbf{t}}^{(\infty)}(2n + 1) = 2\mathcal{P}_{\mathbf{t}}^{(\infty)}(n + 1) \text{ and } \mathcal{P}_{\mathbf{t}}^{(\infty)}(2n) = \mathcal{P}_{\mathbf{t}}^{(\infty)}(n + 1) + \mathcal{P}_{\mathbf{t}}^{(\infty)}(n)$$

for all $n \geq 2$.

Recently there has been a renewal of interest in abelian notions arising in combinatorics on words (e.g., avoiding abelian or ℓ-abelian patterns, abelian bordered words, etc.). For instance, two finite words u and v are *abelian equivalent* if one is obtained by permuting the letters of the other one. Since the Thue–Morse word is an infinite concatenation of factors 01 and 10, this word is *abelian periodic* of period 2. The *abelian complexity* of an infinite word $\mathbf{x}$ is a function $\mathcal{P}_{\mathbf{x}}^{(1)} : \mathbb{N} \to \mathbb{N}$ which maps n to the number of distinct factors of length n occurring in $\mathbf{x}$, counted up to abelian equivalence. Madill and Rampersad [12] provided the first example of regularity in this setting: the abelian complexity of the paper-folding word (which is another typical example of an automatic sequence) is unbounded and 2-regular.

Let $\ell \geq 1$ be an integer. Based on [9] the notions of abelian equivalence and thus abelian complexity were recently extended to ℓ-abelian equivalence and ℓ-abelian complexity [10].

Definition 4. Let u, v be two finite words. We let $|u|_v$ denote the number of occurrences of the factor v in u. Two finite words x and y are *ℓ-abelian equivalent* if $|x|_v = |y|_v$ for all words v of length $|v| \leq \ell$.

As an example, the words 011010011 and 001101101 are 2-abelian equivalent but not 3-abelian equivalent (the factor 010 occurs in the first word but not in the second one). Hence one can define the function $\mathcal{P}_{\mathbf{x}}^{(\ell)} : \mathbb{N} \to \mathbb{N}$ which maps n to the number of distinct factors of length n occurring in the infinite word $\mathbf{x}$, counted up to ℓ-abelian equivalence. In particular, for any infinite word $\mathbf{x}$, we have for all $n \geq 0$

$$\mathcal{P}_{\mathbf{x}}^{(1)}(n) \leq \cdots \leq \mathcal{P}_{\mathbf{x}}^{(\ell)}(n) \leq \mathcal{P}_{\mathbf{x}}^{(\ell+1)}(n) \leq \cdots \leq \mathcal{P}_{\mathbf{x}}^{(\infty)}(n).$$

Since we are interested in ℓ-abelian complexity, it is natural to consider the following operation that permits us to compare factors of length ℓ occurring in an infinite word.

Definition 5. Let $\ell \geq 1$. The *ℓ-block coding* of the word $\mathbf{w} = w_0 w_1 w_2 \cdots$ over the alphabet A is the word

$$\text{block}(\mathbf{w}, \ell) = (w_0 \cdots w_{\ell-1})\,(w_1 \cdots w_\ell)\,(w_2 \cdots w_{\ell+1}) \cdots (w_j \cdots w_{j+\ell-1}) \cdots$$

over the alphabet A^ℓ. If $A = \{0, \ldots, r-1\}$, then it is convenient to identify A^ℓ with the set $\{0, \ldots, r^\ell - 1\}$ and each word $w_0 \cdots w_{\ell-1}$ of length ℓ is thus replaced with the integer obtained by reading the word in base r, i.e., $\sum_{i=0}^{\ell-1} w_i\, r^{\ell-1-i}$. It is well known that the ℓ-block coding of a k-automatic sequence is again a k-automatic sequence [6]. One can also define accordingly the ℓ-block coding of a finite word u of length at least ℓ. For example, the 2-block codings of 011010011 and 001101101 are respectively 13212013 and 01321321, which are abelian equivalent.

Lemma 6. *[10, Lemma 2.3] Let $\ell \geq 1$. Two finite words u and v of length at least $\ell - 1$ are ℓ-abelian equivalent if and only if they share the same prefix (resp. suffix) of length $\ell - 1$ and the words* $\text{block}(u, \ell)$ *and* $\text{block}(v, \ell)$ *are abelian equivalent.*

In this paper, we show that both the period-doubling word $\mathbf{p}$ and the Thue–Morse word $\mathbf{t}$ have 2-abelian complexity sequences which are 2-regular. In [11], the authors studied the asymptotic behavior of $\mathcal{P}_{\mathbf{t}}^{(\ell)}(n)$ and also derived some recurrence relations showing that the abelian complexity $\mathcal{P}_{\mathbf{p}}^{(1)}(n)_{n\geq 0}$ of the period-doubling word $\mathbf{p}$ is 2-regular. From [4], one can deduce some other relations about the abelian complexity of $\mathbf{p}$.

Given the first few terms of a sequence, the second and last authors conjectured the 2-regularity of the sequence $\mathcal{P}_{\mathbf{t}}^{(2)}(n)_{n\geq 0}$ by exhibiting relations that should be satisfied (and proved some recurrence relations for this sequence) [16]. See [2, Section 6] for such a "predictive" algorithm that recognizes regularity. Recently, Greinecker proved the recurrence relations needed to prove the 2-regularity of this sequence [8]. Hopefully, the two approaches are complementary: in this paper, we prove 2-regularity without exhibiting the explicit recurrence relations.

Our approach is based on Theorem 7, which establishes the 2-regularity of a large family of sequences satisfying a recurrence relation with a parameter c and 2^{ℓ_0} initial conditions. Computer experiments suggest that many 2-abelian complexity functions satisfy such a reflection property.

Theorem 7. *Let $\ell_0 \geq 0$ and $c \in \mathbb{Z}$. Suppose $s(n)_{n\geq 0}$ is a sequence such that, for all $\ell \geq \ell_0$ and $0 \leq r \leq 2^\ell - 1$, we have*

$$s(2^\ell + r) = \begin{cases} s(r) + c & \text{if } r \leq 2^{\ell-1} \\ s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}. \end{cases} \tag{1}$$

Then $s(n)_{n\geq 0}$ is 2-regular.

It turns out that the general solution of Equation (1) can be expressed naturally in terms of the sequence $A(n)_{n\geq 0}$ satisfying the recurrence for $\ell_0 = 0$ and $c = 1$ with $A(0) = 0$. The sequence $A(n)_{n\geq 0}$ appears as [14, A007302]. Allouche and Shallit [2] identified this sequence as an example of a regular sequence.

From Equation (1) one can get some information about the asymptotic behavior of the sequence $s(n)_{n\geq 0}$. We have $s(n) = O(\log n)$, and moreover

$$s\left(\frac{4^{\ell+1}-1}{3}\right) = s(4^\ell + \cdots + 4^1 + 4^0) = \left(\ell - \left\lfloor \frac{\ell_0 - 1}{2} \right\rfloor\right) c + s\left(\frac{4^{\lfloor(\ell_0+1)/2\rfloor}-1}{3}\right)$$

for $\ell \geq \lfloor \frac{\ell_0-1}{2} \rfloor$. At the same time, there are many subsequences of $s(n)_{n\geq 0}$ which are constant; for example, $s(2^\ell) = c$ for $\ell \geq \ell_0$.

Example 8. As an illustration of the reflection property described in Theorem 7, we consider in Figure 1 the abelian complexity of the 2-block coding of the period-doubling word $\mathbf{p}$.

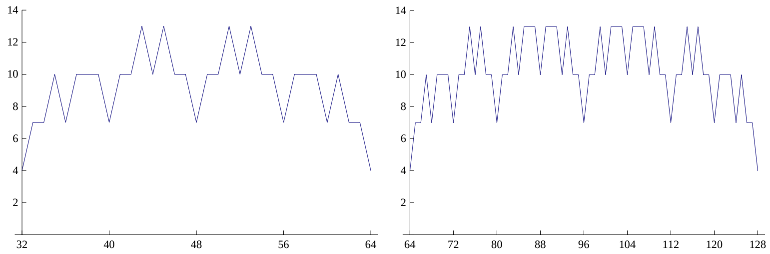

Figure 1: The abelian complexity of $\text{block}(\mathbf{p}, 2)$ on the intervals $[32, 64]$ and $[64, 128]$.

2 2-Abelian complexity of the period-doubling word

To show the 2-regularity of the 2-abelian complexity of $\mathbf{p}$, we consider first the abelian complexity of the 2-block coding $\mathbf{x}$ of $\mathbf{p}$ and then we compare $\mathcal{P}^{(1)}_{\mathbf{x}}(n)$ with $\mathcal{P}^{(2)}_{\mathbf{p}}(n+1)$. The 2-block coding of $\mathbf{p}$ is given by

$$\mathbf{x} := \text{block}(\mathbf{p}, 2) = \phi^{\omega}(1) = 12001212120012001200121212001212\cdots$$

where ϕ is the morphism defined by $\phi : 0 \mapsto 12, 1 \mapsto 12, 2 \mapsto 00$.

We introduce functions related to the number of 0's in the factors of $\mathbf{x}$ of length n. Let $n \in \mathbb{N}$. We let $\max_0(n)$ (resp. $\min_0(n)$) denote the maximum (resp. minimum) number of 0's in a factor of $\mathbf{x}$ of length n. Let $\Delta_0(n) := \max_0(n) - \min_0(n)$ be the difference between these two values.

To prove the 2-regularity of the sequence $\mathcal{P}^{(1)}_{\mathbf{x}}(n)_{n\geq 0}$, we first express $\mathcal{P}^{(1)}_{\mathbf{x}}(n)$ in terms of $\Delta_0(n)$.

Proposition 9. *For $n \in \mathbb{N}$,*

$$\mathcal{P}^{(1)}_{\mathbf{x}}(n) = \begin{cases} \frac{3}{2}\Delta_0(n) + \frac{3}{2} & \text{if } \Delta_0(n) \text{ is odd} \\ \frac{3}{2}\Delta_0(n) + 1 & \text{if } \Delta_0(n) \text{ and } n - \min_0(n) \text{ are even} \\ \frac{3}{2}\Delta_0(n) + 2 & \text{if } \Delta_0(n) \text{ and } n - \min_0(n) + 1 \text{ are even.} \end{cases}$$

To be able to apply the composition result given by Lemma 3 to the expression of $\mathcal{P}^{(1)}_{\mathbf{x}}$, we have therefore to prove that the sequence $\Delta_0(n)_{n\geq 0}$ is 2-regular (this is consequence of the following result) and that the predicates occurring in the previous statement are 2-automatic.

Proposition 10. *Let $\ell \geq 2$ and $0 \leq r < 2^{\ell}$. We have*

$$\Delta_0(2^{\ell} + r) = \begin{cases} \Delta_0(r) + 2 & \text{if } r \leq 2^{\ell-1} \\ \Delta_0(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}. \end{cases}$$

As a consequence of Propositions 9 and 10, $\mathcal{P}^{(1)}_{\mathbf{x}}(n)$ satisfies a reflection recurrence as in Theorem 7 with $\ell_0 = 2$ and $c = 3$. This implies again that the sequence is 2-regular.

Now consider the 2-abelian complexity $\mathcal{P}^{(2)}_{\mathbf{p}}$. To apply Lemma 3, we will express $\mathcal{P}^{(2)}_{\mathbf{p}}$ in terms of the abelian complexity $\mathcal{P}^{(1)}_{\mathbf{x}}$ and the following additional 2-automatic functions.

Definition 11. We define the *max-jump* function $\text{MJ}_0 : \mathbb{N} \to \{0, 1\}$ by $\text{MJ}_0(n) = 1$ when the function $\max_0$ increases. Similarly, let $\text{mj}_0 : \mathbb{N} \to \{0, 1\}$ be the *min-jump* function defined by $\text{mj}_0(n) = \min_0(n+1) - \min_0(n)$.

To compute $\mathcal{P}^{(2)}_{\mathbf{p}}$, we will study when an abelian equivalence class of $\mathbf{x}$ splits into two 2-abelian equivalence classes of $\mathbf{p}$. Let $\mathcal{X}$ be an abelian equivalence class of factors of $\mathbf{x}$ of length n with

n_0 zeros. We can show that $\mathcal{X}$ can possibly lead to two 2-abelian equivalence classes of factors of length $n+1$ of $\mathbf{p}$ only if n and n_0 are both even. In most cases, $\mathcal{X}$ will indeed leads to two distinct 2-abelian equivalence classes. The exceptions can be identified using the max-jump and min-jump functions. The relationship between these two functions and $\mathcal{P}^{(2)}_{\mathbf{p}}$ and $\mathcal{P}^{(1)}_{\mathbf{x}}$ is stated in the following result.

Proposition 12. *Let $n \geq 1$ be an integer. Then*

$$\mathcal{P}^{(2)}_{\mathbf{p}}(n+1) - \mathcal{P}^{(1)}_{\mathbf{x}}(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{\Delta_0(n)}{2} + 1 - \mathrm{MJ}_0(n) - \mathrm{mj}_0(n) & \text{if } n \text{ is even.} \end{cases}$$

In particular, the sequence $\mathcal{P}^{(2)}_{\mathbf{p}}(n)_{n\geq 0}$ is 2-regular.

3 2-Abelian complexity of the Thue–Morse word

In this section, we turn our attention to the Thue–Morse word $\mathbf{t}$. The approach here is similar to the one of the period-doubling word: we consider the abelian complexity of $\mathbf{y} = \mathrm{block}(\mathbf{t}, 2)$, and then we compare $\mathcal{P}^{(1)}_{\mathbf{y}}(n)$ with $\mathcal{P}^{(2)}_{\mathbf{t}}(n+1)$. The 2-block coding of $\mathbf{t}$ is given by

$$\mathbf{y} := \mathrm{block}(\mathbf{t}, 2) = \nu^{\omega}(1) = 132120132012132120121320\cdots$$

where ν is the morphism defined by $\nu : 0 \mapsto 12, 1 \mapsto 13,\ 2 \mapsto 20, 3 \mapsto 21$.

For the Thue–Morse word, the appropriate statistic for factors of $\mathbf{y}$ is the total number of 1's and 2's (or, equivalently, the total number of 0's and 3's). Therefore, for $n \in \mathbb{N}$ we set $\Delta_{12}(n) := \max_{12}(n) - \min_{12}(n)$ where $\max_{12}(n)$ (resp. $\min_{12}(n)$) denote the maximum (resp. minimum) of $\{|u|_1 + |u|_2 : u \text{ is a factor of } \mathbf{y} \text{ with } |u| = n\}$.

In particular, $\Delta_{12}(n)+1$ is the abelian complexity function $\mathcal{P}^{(1)}_{\mathbf{p}}(n)$ of the period-doubling word. This function was also studied in [4, 11]. Here we can obtain relations for Δ_{12} of the same type as in Theorem 7.

As in the previous section, the fact that $\mathcal{P}^{(1)}_{\mathbf{y}}(n)_{n\geq 0}$ is 2-regular will follow from Lemma 3 applied to the next statement.

Proposition 13. *Let $n \in \mathbb{N}$. We have*

$$\mathcal{P}^{(1)}_{\mathbf{y}}(n) = \begin{cases} 2\Delta_{12}(n) + 2 & \text{if } n \text{ is odd} \\ \frac{5}{2}\Delta_{12}(n) + \frac{5}{2} & \text{if } n \text{ and } \Delta_{12}(n)+1 \text{ are even} \\ \frac{5}{2}\Delta_{12}(n) + 4 & \text{if } n,\ \Delta_{12}(n) \text{ and } \min_{12}(n)+1 \text{ are even} \\ \frac{5}{2}\Delta_{12}(n) + 1 & \text{if } n,\ \Delta_{12}(n) \text{ and } \min_{12}(n) \text{ are even.} \end{cases} \quad (2)$$

As in Section 2, we define two new functions $\mathrm{MJ}_{03}(n)$ and $\mathrm{mj}_{03}(n)$ analogously to Definition 11. This permits us to compute the difference $\mathcal{P}^{(2)}_{\mathbf{t}}(n+1) - \mathcal{P}^{(1)}_{\mathbf{y}}(n)$.

Theorem 14. *Let $n \in \mathbb{N}$. The difference $\mathcal{P}^{(2)}_{\mathbf{t}}(n+1) - \mathcal{P}^{(1)}_{\mathbf{y}}(n)$ is equal to*

$$\begin{cases} \Delta_{12}(n) + 2 - 2\,\mathrm{MJ}_{03}(n) - 2\,\mathrm{mj}_{03}(n) & \text{if } n,\ \min_{12}(n)+1 \text{ and } \Delta_{12}(n)+1 \text{ are odd} \\ \Delta_{12}(n) + 1 - 2\,\mathrm{MJ}_{03}(n) & \text{if } n,\ \min_{12}(n)+1 \text{ and } \Delta_{12}(n) \text{ are odd} \\ \Delta_{12}(n) + 1 - 2\,\mathrm{mj}_{03}(n) & \text{if } n,\ \min_{12}(n) \text{ and } \Delta_{12}(n) \text{ are odd} \\ \Delta_{12}(n) & \text{if } n,\ \min_{12}(n) \text{ and } \Delta_{12}(n)+1 \text{ are odd} \\ \frac{1}{2}\Delta_{12}(n) + 1 & \text{if } n,\ \min_{12}(n) \text{ and } \Delta_{12}(n) \text{ are even} \\ \frac{1}{2}\Delta_{12}(n) & \text{if } n,\ \min_{12}(n)+1 \text{ and } \Delta_{12}(n) \text{ are even} \\ \frac{1}{2}\Delta_{12}(n) + \frac{1}{2} & \text{if } n \text{ and } \Delta_{12}(n)+1 \text{ are even.} \end{cases}$$

In particular, the sequence $\mathcal{P}^{(2)}_{\mathbf{t}}(n)_{n\geq 0}$ is 2-regular.

4 Conclusions

The two examples treated in this paper suggest that a general framework to study the ℓ-abelian complexity of k-automatic sequences may exist. Indeed, one conjectures that *any k-automatic sequence has an ℓ-abelian complexity function that is k-regular*. As an example, if we consider the 3-block coding of the period-doubling word,

$$\mathbf{z} = \text{block}(\mathbf{p}, 3) = 240125252401240124 \cdots .$$

The abelian complexity $\mathcal{P}^{(1)}_{\mathbf{z}}(n)_{n\geq 0} = (1, 5, 5, 8, 6, 10, 19, 11, \ldots)$ seems to satisfy, for $\ell \geq 4$, the following relations (which are quite similar to what we have discussed so far)

$$\mathcal{P}^{(1)}_{\mathbf{z}}(2^\ell + r) = \begin{cases} \mathcal{P}^{(1)}_{\mathbf{z}}(r) + 5 & \text{if } r \leq 2^{\ell-1} \text{ and } r \text{ even} \\ \mathcal{P}^{(1)}_{\mathbf{z}}(r) + 7 & \text{if } r \leq 2^{\ell-1} \text{ and } r \text{ odd} \\ \mathcal{P}^{(1)}_{\mathbf{z}}(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}. \end{cases}$$

Then, the second step would be to relate $\mathcal{P}^{(3)}_{\mathbf{p}}$ with $\mathcal{P}^{(1)}_{\mathbf{z}}$.

References

[1] Jean-Paul Allouche and Jeffrey Shallit, The ring of k-regular sequences, *Theoretical Computer Science* **98** (1992) 163–197.

[2] Jean-Paul Allouche and Jeffrey Shallit, The ring of k-regular sequences II, *Theoretical Computer Science* **307** (2003) 3–29.

[3] Jean-Paul Allouche and Jeffrey Shallit, *Automatic sequences. Theory, applications, generalizations*, Cambridge University Press, Cambridge, 2003.

[4] Francine Blanchet-Sadri and James D. Currie and Narad Rampersad and Nathan Fox, Abelian complexity of fixed point of morphism $0 \mapsto 012, 1 \mapsto 02, 2 \mapsto 1$, *INTEGERS* **14** (2014) #A11.

[5] Émilie Charlier and Narad Rampersad and Jeffrey Shallit, Enumeration and decidable properties of automatic sequences, *Internat. J. Found. Comput. Sci.* **23** (2012), no. 5, 1035–1066.

[6] Alan Cobham, Uniform tag sequences, *Math. Systems Theory* **6** (1972), 186–192.

[7] Samuel Eilenberg, *Automata, Languages and Machines*, Vol. A, Academic Press, New York, 1974.

[8] Florian Greinecker, On the 2-abelian complexity of Thue–Morse subwords, `arXiv:1404.3906`.

[9] Juhani Karhumäki, Generalized Parikh mappings and homomorphisms, *Information and Control* **47** (1980), 155–165.

[10] Juhani Karhumäki and Aleksi Saarela and Luca Q. Zamboni, On a generalization of Abelian equivalence and complexity of infinite words, *J. Combin. Theory Ser. A* **120** (2013), no. 8, 2189–2206.

[11] Juhani Karhumäki and Aleksi Saarela and Luca Q. Zamboni, Variations of the Morse-Hedlund theorem for k-abelian equivalence, `arXiv:1302.3783v1`.

[12] Blake Madill and Narad Rampersad, The abelian complexity of the paperfolding word *Discrete Math.* **313** (2013), no. 7, 831–838.

[13] Brigitte Mossé, Reconnaissabilité des substitutions et complexité des suites automatiques, *Bull. Soc. Math. France* **124** (1996), 329–346.

[14] The OEIS Foundation, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

[15] Aline Parreau, Michel Rigo, Eric Rowland and Élise Vandomme, A new approach to the 2-regularity of the ℓ-abelian complexity of 2-automatic sequences, `arXiv:1405.3532`.

[16] Michel Rigo and Élise Vandomme, 2-abelian complexity of the Thue–Morse sequence, `http://hdl.handle.net/2268/135841`, December 2012.