A note on regular interval exchange sets over a quadratic field

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Abstract

A set of words is said *morphic* if it is the set of factors of a morphic word. We prove that the interval exchange set relative to a regular interval exchange transformation defined over a quadratic field is morphic. We give two different proofs of this result, the first using the notion of return words and derived words, and the second employing only some properties of interval exchange transformations.

1 Introduction

Interval exchange transformations were introduced by Oseledec [19] following an earlier idea of Arnol'd [1]. These transformations form a generalization of rotations of the circle (the two notions coincide when there are exactly 2 intervals). Rauzy introduced in [21] a transformation, now called Rauzy induction (or Rauzy-Veech induction), which operates on interval exchange transformations. It transforms an interval exchange transformation into another, operating on a smaller semi-interval. Its iteration can be viewed as a generalization of the continued fraction development. The induction consists of taking the first return map of the transformation with respect to a particular subsemi-interval of the original semi-interval. A two-sided version of the Rauzy induction is studied in [6], along with a characterization of the so-called *admissible semi-intervals*, namely the semi-intervals reachable by the iteration of this two-sided induction.

Interval exchange transformations defined over quadratic fields are studied by Boshernitzan and Carroll in [8] and [9]. They show that, using iteratively the first return map on one of the semi-intervals exchanged by this kind of transformation, one obtains only a finite number of different new transformations up to rescaling. This result extends Lagrange's classical theorem saying that quadratic irrationals have a periodic continued fraction expansion. In this paper we generalize this result by studying interval exchange transformations defined over quadratic fields and inducing on two-sided admissible semi-intervals (see also [6]).

Several authors have studied the links between combinatorics on words and dynamical systems (for example [15]). For the particular case of interval exchange transformations, see for example [2], [6], [13] and [16]. The main contribution of this paper is that a regular interval exchange set, i.e. the set of factors of a natural coding of a regular interval exchange transformation, is the set of factors of a primitive morphic word (Theorem 7).

The paper is organized as follows. Section 2 is devoted to some basic definitions of combinatorics on words. In Section 3, we recall some notions concerning interval exchange transformations, such as minimality and regularity. We also introduce an equivalence relation on the set of interval exchange transformations. In Section 4 we evoke the Rauzy induction and the generalization to its two-sided version. We also recall the definition of admissibility and show how this notion is related to the Rauzy induction (Theorem 1). We conclude the section by introducing the equivalence graph of a regular interval exchange transformation. Section 5 is devoted to the natural coding of an interval exchange transformation and the related notion of interval exchange set, i.e. the set of factors of a natural coding. We recall a result of [6] showing that two particular families of semi-intervals are admissible (Proposition 3) and that two equivalent regular interval exchange transformations, with the second obtained from the first by a sequence of Rauzy inductions, have the same set of factors. The final part of this paper, Section 6, is devoted to the proof of our main contribution (Theorem 7). We give two different demonstrations of this result. First, we use the notion of derived words and return words, as well as Durand's result (see [12]) on derived words and primitive morphic words. Thereafter, we show the same result using only the properties of minimal and regular interval exchange transformations.

2 Words and sets

Let A be a finite nonempty alphabet. All words considered below, unless stated explicitly, are supposed to be on the alphabet A. We denote by A^* the set of all words on A. We denote by 1 or by ε the empty word. We refer to [3] for the notions of a prefix, suffix and factor of a word. See also [7] for a more detailed presentation of words, sets and morphisms.

A *morphism* $f : A^* \to B^*$ is a monoid morphism from A^* into B^*. If $f : A^* \to A^*$ and $a \in A$ is such that the word $f(a)$ begins with a and if $|f^n(a)|$ tends to infinity with n, there is a unique infinite word denoted $f^\omega(a)$ which has all words $f^n(a)$ as prefixes. It is called a *fixpoint* of the morphism f. A morphism $f : A^* \to A^*$ is called *primitive* if there is an integer k such that for all $a, b \in A$, the letter b appears in $f^k(a)$. An infinite word y over an alphabet B is called *morphic* if there exists a morphism f on an alphabet A, a fixpoint $x = f^\omega(a)$ of f and a morphism $\sigma : A^* \to B^*$ such that $y = \sigma(x)$. If $A = B$ and σ is the identity map, we call y *purely morphic.* If f is primitive we say that the word is *primitive morphic.*

A set F of words on the alphabet A is said to be *factorial* if it contains the factors of its elements. A factorial set of words $F \neq \{1\}$ is *recurrent* if for every $u, w \in F$ there exists a $v \in F$ such that $uvw \in F$. If F is a recurrent set, then there exists an infinite word x such that $F = F(x)$, i.e. F is the set of factors of a word x (see, for example, [3]). Extending the definition, we say that a set $F(x)$ is *morphic* (resp. *purely morphic*, *primitive morphic*) if the infinite word x is morphic (resp. purely morphic, primitive morphic). A set F is said to be *right-extendable* if for every $w \in F$ there exists some $a \in A$ such that $wa \in F$. It is said to be *uniformly recurrent* if it is right-extendable and if, for any word $u \in F$, there exists an integer $n \geq 1$ such that u is a factor of every word of F of length n. If f is a primitive morphism, the set of factors of any fixpoint of f is uniformly recurrent (see [14]). It is easy to see that uniform recurrence implies recurrence.

3 Interval exchange transformation

Let $[\ell, r[$ be a nonempty semi-interval of the real line and $A = \{a_1, a_2, \ldots, a_s\}$ a finite ordered alphabet with $a_1 < a_2 < \cdots < a_s$. Let $(I_a)_{a\in A}$ be an ordered partition of $[\ell, r[$ into semi-intervals and denote by λ_i the length of I_{a_i}. Let $\pi \in \mathcal{S}_s$ be a permutation on A. We define $\gamma_i = \sum_{a_j<a_i} \lambda_j$ and $\delta_i = \sum_{\pi(a_j)<\pi(a_i)} \lambda_j$. The *interval exchange transformation* relative to $(I_a)_{a\in A}$ is the map $T : [\ell, r[\to [\ell, r[$ defined by

$$T(z) = z + \alpha_a \quad \text{if } z \in I_a,$$

where $\alpha_a = \delta_a - \gamma_a$. The values $(\alpha_a)_{a\in A}$ are called the *translation values* of T. Observe that the restriction of T to I_a is a translation onto $J_a = T(I_a)$, that γ_i is the left boundary of I_{a_i} and that δ_j is the left boundary of J_{a_i}. Note that the family $(J_a)_{a\in A}$ is also a partition of $[\ell, r[$. In particular, the transformation T defines a bijection from $[\ell, r[$ onto itself. An interval exchange transformation relative to $(I_a)_{a\in A}$ is also said to be an s-interval exchange transformation. We will also denote $T = T_{\pi,\lambda}$, where $\lambda = (\lambda_i)_{a_i\in A}$ is the ordered sequence of the lengths of the semi-intervals. Note that the transformation $T_{\pi,\lambda}$ does not depend on the choice of the left point ℓ.

To give an example, every rotation of angle α on the semi-interval $[0, 1[$, i.e. the function defined as $T(z) = z + \alpha \bmod 1$ for every $0 \le z < 1$, is an interval exchange transformation relative to the partition $([0, 1-\alpha[, [1-\alpha, 1[)$ with permutation $\pi = (12)$.

The *orbit* of a point $z \in [\ell, r[$ is the set $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$. The transformation T is said to be *minimal* if for any $z \in [\ell, r[$, $\mathcal{O}(z)$ is dense in $[\ell, r[$. The points $0 = \gamma_1, \gamma_2, \ldots, \gamma_s$ form the set of *separation points* of T, denoted $\mathrm{Sep}(T)$. Note that the transformation T has at most $s-1$ *singularities* (points at which it is not continuous), which are among the nonzero separation points $\gamma_2, \ldots, \gamma_s$. An interval exchange transformation $T_{\pi,\lambda}$ is called *regular* if the orbits of the nonzero separation points $\gamma_2, \ldots, \gamma_s$ are infinite and disjoint. Note that the orbit of 0 cannot be disjoint from the others since one has $T(\gamma_i) = 0$ for some i with $2 \le i \le s$. A regular interval exchange transformation is also said to satisfy the *idoc* (infinite disjoint orbit condition). As an example, every rotation of irrational angle is a regular 2-interval exchange transformation. Any regular interval exchange transformation is minimal (see [18]).

Two s-interval exchange transformations $T_{\pi,\lambda}$ and $T_{\sigma,\mu}$ are said to be *equivalent* if $\sigma = \pi$ and $\mu = c\lambda$ for some $c > 0$. We denote by $[T_{\pi,\lambda}]$ the equivalence class of $T_{\pi,\lambda}$.

4 Rauzy induction

Let $T = T_{\pi,\lambda}$ be a minimal s-interval exchange transformation on $[\ell, r[$ and let $I \subset [\ell, r[$ be a semi-interval. Since T is minimal, for each $z \in [\ell, r[$ there exists an integer $n > 0$ such that $T^n(z) \in I$. The *transformation induced* by T on I is the first return map on I, i.e. the transformation $S : I \to I$ defined for $z \in I$ by $S(z) = T^n(z)$ with $n = \min\{n > 0 \mid T^n(z) \in I\}$. The semi-interval I is called the *domain* of S, denoted $D(S)$. Note that the transformation induced by an s-interval exchange transformation on $[\ell, r[$ on any semi-interval included in $[\ell, r[$ is always an interval exchange transformation on at most $s + 2$ intervals (see [10]).

Rauzy introduced in [21] a transformation ψ, now called *right Rauzy induction*, which operates on interval exchange transformations. It transforms an interval exchange transformation into another, operating on a smaller semi-interval. Namely, $\psi(T)$ is the transformation induced by T on $[\ell, \max\{\gamma_s, \delta_{\pi(s)}\}[$. In the same paper, the intervals obtained by iteration of this induction, the so-called *right admissible* semi-intervals, are characterized. It is also proved that the Rauzy induction preserves the regularity if the domain is a right-admissible semi-interval.

The symmetrical notion of *left Rauzy induction*, denoted by φ, is defined in [6], where symmetrical results are also shown. In the same paper a two-sided version of the Rauzy induction is introduced along with the generalized notion of *admissible* semi-interval.

One of the main results of [6] is the following:

Theorem 1 *Let T be a regular s-interval exchange transformation on $[\ell, r[$. For any admissible semi-interval I, the transformation S induced by T on I is a regular s-interval exchange transformation. Moreover, a semi-interval I is admissible for T if and only if there exists a sequence $\chi \in \{\varphi, \psi\}^*$ such that I is the domain of $\chi(T)$. In this case, the transformation induced by T on I is $\chi(T)$.*

For every $a \in A$, the semi-intervals I_a and J_a are admissible (see [6]). In the following we show an important generalization of this result (Proposition 3).

For an interval exchange transformation T we consider the labelled directed graph $G(T)$, called the *equivalence graph* of T, defined as follows. The vertices are the equivalence classes of transformations obtained starting from T and applying all possible $\chi \in \{\psi, \varphi\}^*$. There is an edge labelled ψ starting from a vertex $[T]$ to a vertex $[S]$ if and only if $S = \psi(T)$ for two transformations $T \in [T]$ and $S \in [S]$. We define similarly the edges labelled φ.

Note that, in general, the equivalence graph can be infinite. A sufficient condition for the equivalence graph to be finite is proved in [11] (generalizing a result of [9]). Namely, the graph is finite if the lengths of all exchanged intervals belong to a quadratic number field. We can reformulate this as follows:

Theorem 2 *Let T be a regular interval exchange transformation defined over a quadratic field. The family of all induced transformation of T over an admissible semi-interval contains finitely many distinct transformations up to equivalence.*

5 Natural coding

Let T be an interval exchange transformation relative to $(I_a)_{a\in A}$. For a given point $z \in [\ell, r[$, the *natural coding* of T relative to z is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots$ on the alphabet A defined by

$$a_n = a \quad \text{if} \quad T^n(z) \in I_a.$$

For a word $w = b_0 b_1 \cdots b_{m-1}$, let I_w be the set

$$I_w = I_{b_0} \cap T^{-1}(I_{b_1}) \cap \ldots \cap T^{-m+1}(I_{b_{m-1}}).$$

Set $J_w = T^m(I_w)$. Thus

$$J_w = T^m(I_{b_0}) \cap T^{m-1}(I_{b_1}) \cap \cdots \cap T(I_{b_{m-1}}).$$

In particular, we have $J_a = T(I_a)$ for $a \in A$. Note that, for every $w \in A^*$, both I_w and J_w are semi-intervals (see [4]). We set by convention $I_\varepsilon = J_\varepsilon = [\ell, r[$. Then one has for any $n \geq 0$

$$a_n a_{n+1} \cdots a_{n+m-1} = w \Longleftrightarrow T^n(z) \in I_w$$

and

$$a_{n-m} a_{n-m+1} \cdots a_{n-1} = w \Longleftrightarrow T^n(z) \in J_w.$$

Let $(\alpha_a)_{a\in A}$ be the translation values of T. Note that for any word $w \in A^*$, $J_w = I_w + \alpha_w$, where $\alpha_w = \sum_{j=0}^{m-1} \alpha_{b_j}$ (see [4]). In particular, the restriction of $T^{|w|}$ to I_w is a translation.

If T is a minimal interval exchange transformation, one has $w \in F(\Sigma_T(z))$ if and only if $I_w \neq \emptyset$. Thus the set $F(\Sigma_T(z))$ does not depend on z. Since it depends only on T, we denote it by $F(T)$. When T is regular (resp. minimal), such a set is called a *regular interval exchange set* (resp. a *minimal interval exchange set*). The following result is proved in [6].

Proposition 3 *Let T be a regular interval exchange transformation. For any $w \in F(T)$, the semi-intervals I_w and J_w are admissible.*

Let T be a regular interval exchange transformation and S a transformation obtained from T by two-sided Rauzy inductions. The following theorem, proved in [6], proves that the set of factors of T is the same of the set of factors of S up to isomorphism.

Theorem 4 *Let T be a regular interval exchange transformation. For $\chi \in \{\psi, \varphi\}^*$, let $S = \chi(T)$ and let I be the domain of S. There is an automorphism θ of the free group on A such that $\Sigma_T(z) = \theta(\Sigma_S(z))$ for all $z \in I$.*

Note that if the transformations T and $S = \chi(T)$, with $\chi \in \{\psi, \varphi\}^*$, are equivalent, then there exists a point $z_0 \in I$ such that z_0 is a fixpoint of the isometry that transforms $D(S)$ into $D(T)$ (if χ is different from the identity map, this point is unique). In that case one has $\Sigma_S(z_0) = \Sigma_T(z_0) = \theta\left(\Sigma_S(z_0)\right)$ for an appropriate automorphism θ, that is $\Sigma_T(z_0)$ is a fixpoint of an appropriate automorphism.

Let $T = T_{\pi,\lambda}$. The automorphism θ of the free group defined in Theorem 4 is obtained as a composition of the two elementary automorphism θ_1, θ_2, extentions of the monoidal morphism from A^* into itself defined by

$$\theta_1(a) = \begin{cases} a_{\pi(s)} a_s & \text{if } a = a_{\pi(s)} \\ a & \text{otherwise} \end{cases}, \quad \theta_2(a) = \begin{cases} a_{\pi(s)} a_s & \text{if } a = a_s \\ a & \text{otherwise} \end{cases}.$$

This automorphism "keeps track" of the iteration of T in the first return map S, by mapping every letter a of $\Sigma_S(z)$ to the word $\theta(a) \in F(\Sigma_T(z))$, corresponding to the "path" of z in the natural coding of T (see [6] for a more detailed presentation and examples).

6 Return words and derived words

Let F be a recurrent set. For $w \in F$, let $\Gamma_F(w) = \{x \in F \mid xw \in F \cap wA^+\}$ be the set of *left return words* to w and let $\mathcal{R}_F(w) = \Gamma_F(w) \setminus A^+\Gamma_F(w)$ be the set of *first left return words* to w. Clearly, a recurrent set F is uniformly recurrent if and only if the set $\mathcal{R}_F(w)$ is finite for any $w \in F$.

Let F be a recurrent set and let $w \in F$. A *coding morphism* for the set $\mathcal{R}_F(w)$ is a morphism $f_w : B^* \to A^*$ which maps bijectively the (possibly infinite) alphabet B onto $\mathcal{R}_F(w)$ (note that this morphism is unique up to renaming the letters of B). The set $f_w^{-1}(Fw^{-1})$, denoted $D_w(F)$, is called the *derived set* of F with respect to w.

Let F be a recurrent set and x be an infinite word such that $F = F(x)$. Let $w \in F$ and let f_w a coding morphism for the set $\mathcal{R}_F(w)$. Since w appears infinitely often in x, there is a unique factorization $x = vz$ with $z \in \mathcal{R}_F(w)^\omega$ where w is not a factor of v. Clearly, if w is a prefix of x then $v = \varepsilon$. The infinite word $f_w^{-1}(z)$ is called the *derived word* of x relative to w, denoted $\mathcal{D}_w(x)$. The derived set of F with respect to w is the set of factors of the derived words of x relative to w, that is $\mathcal{D}_w(F) = F\left(\mathcal{D}_w(x)\right)$ (see [6]).

It is proved in [6] that the family of regular interval exchange sets is closed under derivation, meaning that any derived set of a regular s-interval exchange set is a regular s-interval exchange set. The same property holds for Sturmian sets (see [17]) and for uniformly recurrent tree sets (see [5]).

Let T be a regular interval exchange transformation relative to a partition of semi-intervals whose length belongs to a quadratic number field. Let $z \in D(T)$. Suppose that $\Sigma_T(z) = b_0 b_1 \cdots$. Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of interval exchange transformations defined as $T_0 = T$, and T_{n+1} the transformation induced by T_n on the interval $J_{b_0 b_1 \cdots b_n}$. Since J_w is admissible for every $w \in F(T)$, every transformation of the sequence is obtained by the previous transformation using iteratevely the two-sided Rauzy induction. Hence, by Theorem 2, the sequence $(T_n)_n$ contains finitely many distinct transformations up to equivalence. Let $k \geq h \geq 0$ be such that $[T_h] = [T_k]$. Then it is easy to see that the family of distinct transformations is exactly

$\{T_0, T_1, \dots T_{k-1}\}$. Moreover, by Theorem 4 and the remark following it, there exists a point $z_0 \in D(T_k)$ such that $\Sigma_{T_{h+p}}(z_0) = \Sigma_{T_{k+p}}(z_0)$ for every $p \geq 0$.

Set $x = \Sigma_T(z_0) = b_0 b_1 \cdots$. By the observation at the end of Section 5, one has $\mathcal{D}_{b_0 b_1 \cdots b_n}(x) = \Sigma_{T_n}(z_0)$. Hence, we have the following:

Proposition 5 *Let T be a regular interval exchange transformation. There exists a point $z \in D(T)$ such that the number of derived words of $\Sigma_T(z)$ relative to its prefixes is finite.*

In [12], F. Durand proved the following result, linking derived words to primitive morphic words.

Theorem 6 *An infinite word is primitive morphic if and only if the number of its different derived words relative to its prefixes is finite.*

From Theorem 6 and Proposition 5 it easily follows that, given a regular interval exchange transformation T, there exists a point $z \in D(T)$ such that the natural coding of T relative to z is primitive morphic. Since T is minimal, the set $F(T)$ does not depend on the point z. Therefore, we have the following result:

Theorem 7 *Let T be a regular interval exchange transformation defined over a quadratic field. Then the interval exchange set $F(T)$ is primitive morphic.*

We conclude the section by giving an alternative and direct proof of Theorem 7, without using the notion of derived words. In order to do this, we need some preliminar results.

Lemma 8 *Let T be a minimal interval exchange transformation. For every $N > 0$ there exists an $\varepsilon > 0$ such that for every $z \in D(T)$, one has*

$$|T^n(z) - z| < \varepsilon \quad \Longrightarrow \quad n \geq N.$$

Sketch of the proof. We can choose $\varepsilon = \min\left\{ \left| \sum_{i_j=1}^{M} \alpha_{i_j} \right| \mid i_j = 1, \dots s,\ M \leq N \right\}$. ∎

Lemma 9 *Let $T, \chi(T)$ be two equivalent regular interval exchange transformations with $\chi \in \{\varphi, \psi\}^*$. There exists a primitive morphism θ and a point $z \in D(T)$ such that the natural coding of T relative to z is a fixpoint of θ.*

Proof. Every natural coding of T is uniformly recurrent (see [4]). Thus, there exists a positive integer N such that every letter of the alphabet appears in every word of length N of $F(T)$. Moreover, applying iteratively the Rauzy induction, the length of the domains tends to 0 (see Theorem 3.12 of [6]). Consider $T' = \chi^m(T)$, for a big enough positive integer m, such that $D(T') < \varepsilon$, where ε is the positive real number for which, by Lemma 8, the first return map for every point of the domain is "longer" than N, i.e. $T'(z) = T^{n(z)}(z)$, with $n(z) \geq N$, for every $z \in D(T')$. By Theorem 4 and the remark following it, there exists an automorphism θ of the free group and a point $z \in D(T)$ such that the natural coding of T relative to z is the image by θ of the natural coding of T' relative to the same point, i.e. $\Sigma_T(z) = \theta\left(\Sigma_{T'}(z)\right)$. By the previous argument, the image of every letter by θ is longer than N, hence it contains every letter of the alphabet as a factor. Therefore, θ is a primitive morphism. ∎

Using the previous lemmas we can give an alternative proof of Theorem 7.

Alternative proof of Theorem 7. By Theorem 2 there exists a regular interval transformation S such that we can find in the equivalence graph of T a path from $[T]$ to $[S]$ followed by a cycle on $[S]$. Thus, there exists a point $z \in D(S)$ and two automorphisms θ, η of the free group such that $\Sigma_T(z) = \theta\left(\Sigma_S(z)\right)$, with $\Sigma_S(z)$ a fixpoint of η. By Lemma 9 we can suppose, without loss of generality, that η is primitive. Therefore, $F(T)$ is a primitive morphic set. ∎

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