Statistics of large binary sequences

Jordan Emme[*] and Alexander Prikhod'ko[†]

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Abstract

In this talk we will present a study of the equation $[x] = [x + a] + s$ where x and a are positive integers, s is an integer and $[x]$ denotes the number of "1" in the binary decomposition of x. We will be interested in solving this equation for fixed a a s as well as the statistical behaviour of $[x] - [x + a]$ for a fixed positive integer a.

1 Introduction

Let x be a positive integer and $[x]$ denote the number of "1" in the binary expansion of x. We are interested in solving the equation $[x] = [x + a] + s$ for fixed $a \in \mathbb{N}$ and $s \in \mathbb{Z}$. The means employed for such a problem are mainly combinatorical via the construction of a summation tree. Knowing the structure of solutions of such an equations allows, for each $a \in \mathbb{N}$, the study of the distribution of probability of the difference $[x] - [x + a]$, given by the function μ_a over $l^1(\mathbb{Z})$, where x can be identified with its binary expansion and so as a sequence of 0 and 1 with balanced Bernouilli distribution of probability. To this end, we study further the summation tree we introduced earlier. Being able to compute such a probabilty measure for each positive integer a we then focus on the study of its asymptotic behaviour as a gets large. This involves looking at pathes in a particular Schreier graph of the Baumslag-Solitar group of type $(1, 2)$.

2 Results

We wish to have a precise, constructive, understanding of the solutions of the equation $[x] = [x + a] + s$ for any set of parameters a and s. To this end, we construct an infinite binary tree associated to a on which it is possible to read the binary expansion of solutions to this equation as pathes on this tree. An example of a part of such a tree is given on figure 1. Such a construction allows us to prove the following theorem :

Theorem 1 *Let $a \in \mathbb{N}$ and $s \in \mathbb{Z}$. There exists a finite set of prefixes*

$$\mathcal{P} = \{p_1, ..., p_k\} \subset \{0, 1\}^*$$

such that $x \in \mathbb{N}$ is solution of $[x] = [x + a] + s$ if and only if the binary expansion of x starts with one of the prefixes p_i.

[*]Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France. E-mail : jordan.emme@univ-amu.fr

[†]Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France. E-mail : sasha.prihodko@gmail.com

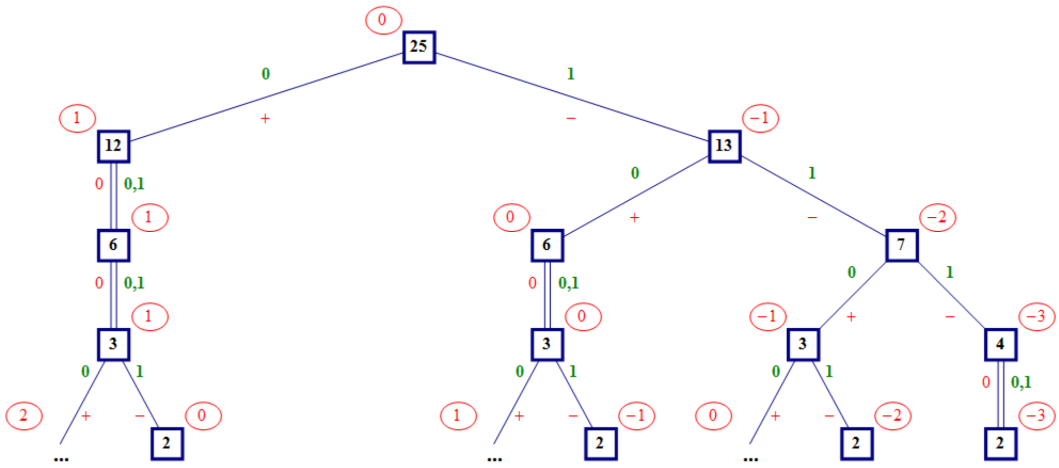

Figure 1: Part of the tree for $a = 25$

Let us now define, for any positive integer a, the function $\mu_a \in l^1(\mathbb{Z})$ defined by

$$\forall s \in \mathbb{Z}, \quad \mu_a(s) = \mathbb{P}(\{x \in \mathbb{N} \quad | \quad [x] - [x+a] = s\})$$

where $\mathbb{P}$ is the balanced Bernouilli probability measure on $\{0,1\}^*$ and by identifying the integer x and its binary expansion. Collapsing the tree on a particular Bratelli diagram as shown in figure 2 and understanding its patterns allows us to prove the next theorem :

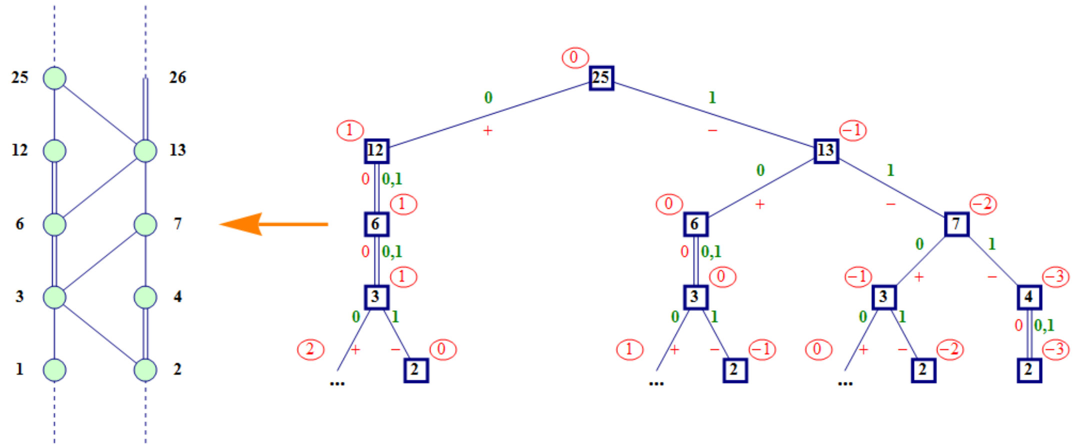

Figure 2: Collapsing the adding tree for $a = 25$.

Theorem 2 *The function μ_a is calculated via an infinite product of matrices*

$$\mu_a = (1,\ 1)\ \cdots A_{a_n} A_{a_{n-1}} \cdots A_{a_1} A_{a_0} \begin{pmatrix} \delta_0 \\ 0 \end{pmatrix},$$

where $a = a_0 + 2a_1 + \dots$ is a binary expansion of a, $\delta_0(i) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ 1 & \text{otherwise} \end{cases}$

$$A_0 = \begin{pmatrix} Id & \frac{1}{2}\hat{\sigma} \\ 0 & \frac{1}{2}\hat{\sigma}^{-1} \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \frac{1}{2}\hat{\sigma} & 0 \\ \frac{1}{2}\hat{\sigma}^{-1} & Id \end{pmatrix},$$

and $\hat{\sigma} : (p_j) \mapsto (p_{j+1})$ is the shift transformation on $l^1(\mathbb{Z})$.

The final part of our investigation is dedicated to the asymptotic behaviour of μ_a for large integers a. We have to use the following object :

Definition 1 *Let G be a finitely generated group with a generator set $\mathcal{S}$, and let H be a subgroup, not necessary normal, such that $S \cap H = \emptyset$. The Schreier graph for the triple $(G, H, \mathcal{S})$ is defined as the orientated graph with vertex set G/H and edge set $E = \{(aH, saH) | a \in G,\ s \in \mathcal{S}\}$.*

The group we wish to consider is the Baumslag-Solitar group of type (1,2) which is defined by

$$BS(1,2) = \langle \sigma, S \quad | \quad \sigma S \sigma^{-1} = S^2 \rangle$$

in the particular case where the generators are the following real functions

$$\sigma : y \to 2y, \qquad S : y \to y + 1.$$

This group naturally acts on the set of diadic integers. Then, for the generator set $\{S, S^{-1}, \sigma\}$ and a certain subgroup of $BS(1,2)$, there is an associated Schreier graph Γ where it is possible to associate vertices to diadic integers.
Then, for all integer a, denote by γ_a the geodesic linking 0 to a in Γ and denote by w the weight function on $BS(1,2)$ taking value 1 on S, S^{-1} and 0 on σ, σ^{-1}. Finally, let $\|a\|_0 = \int_{\gamma_a} w(g)dg$. We have the following result :

Theorem 3 *For $\|a\|_0$ large enough, we have the following inequality:*

$$\|\mu_a\|_2 \leq C_0 \cdot \|a\|_0^{-1/4}$$

where C_0 is a universal constant.

The study of such a problem is motivated by its links with some ergodic properties of Vershik's transformation in the Pascal triangle.