Stability analysis of discrete time switching systems driven by an automaton.*

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Abstract

In this work, we develop a framework for stability analysis of Discrete-Time Linear Switching Systems (DTLSS), a class of hybrid dynamical system mixing continuous and discrete state dynamics. We focus on the finite *switching automaton* describing the discrete state dynamics of the system. First, we study a quantity depicting the worst-case growth rate of the system, the *constrained joint-spectral radius ρ*. We relate it with stability and describe how the switching automaton influences it. Then we propose an approximation scheme for ϵ-relative approximation of ρ based on a powerful mathematical construction we call *nodal multi-norms*.

1 Discrete-time linear switching systems

A discrete-time switching linear system (DTLSS) is a dynamical system of the form

$$x_{t+1} = A_{\sigma(t)} x_t$$
$$x_0 \in \mathbb{R}^n.$$

At time t, a matrix $A_{\sigma(t)}$ is selected amongst a set of N matrices $\Sigma = \{A_1, \cdots, A_N\}$. The value of $\sigma(t) \in \{1, \cdots, N\}$, is called the *mode* of the system at time t. The succession of modes up to time t, $(\sigma(0), \cdots, \sigma(t)) \in \{1, \cdots, N\}^t$ is referred to as the *switching sequence* of the system up to time t.

For some systems, there might be additional informations about the shape of the switching sequences. Our focus is on using this additional information in developing stability conditions for the switching system.

When the switching is either *arbitrary* or unknown, a quantity known as the joint-spectral radius (see [5]) has been developed to answer the question. It is a real positive number, corresponding to the worst case growth of any trajectory of the system.

When it is known that the switching is not arbitrary, sufficient stability conditions based on the joint spectral radius might be too conservative, as illustrated in Example 1.1.

Example 1.1. *Consider the set $\Sigma = \{A_1, A_2, A_3, A_4\}$, with $0 < \epsilon < 1$ and*

$$A_1 = \begin{pmatrix} 0.99 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1-\epsilon \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & \frac{1-\epsilon}{2} \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \frac{2}{1-\epsilon} & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Under arbitrary switching, the system is unstable since A_4 has a spectral radius $\frac{2}{1-\epsilon}$ greater than 1. In this simple example, this corresponds to the joint-spectral radius of the system.

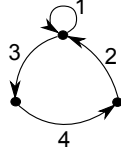*Now consider the switching is made to correspond to transitions in the automaton illustrated in Figure 1.*

Figure 1: Language-graph for the matrix set (1).

The system is now stable. Indeed, $A_1 A_2 = 0.99 A_2$, $A_3 A_2 = 0$, $A_2 A_4 = 0$. Therefore, the worst-case dynamics repeat matrix A_1 an infinite amount of time, with a spectral radius lower than 1.

As pointed out above, we will consider restrictions on the possible switching sequences by seeing them as sequences of transitions in a labelled automaton called the *switching automaton*.

The proofs are omitted due to space constraints.

2 Stability under constrained switching

We now properly define the dynamical systems we are interested in.

Definition 2.1. *A constrained switching system $S(G, \Sigma)$ is a dynamical system composed of a switching automaton represented by a directed labeled graph $G(V, E)$, and a set of N matrices $\Sigma = \{A_1, \cdots, A_N\}$. The dynamics are given by*

$$\begin{aligned} x_{t+1} &= A_{\sigma(t)} x_t \\ x_0 &\in \mathbb{R}^n. \end{aligned} \tag{2}$$

where for all $t \geq 0$, $(\sigma(0), \cdots, \sigma(t-1))$ is a sequence of labels corresponding to a path of length t in the graph G.

In the following, we will refer to switching automata directly through their labeled directed graph representation, $G(V, E)$. In this graph, the nodes V represent states of an automaton, while each edge $e = \{v_i, v_j, k\} \in E$ represents a transition in the automaton from v_i to v_j with label k. In the context of the dynamical systems described above, a path in the graph is an infinite sequence of edges $(\{v(0), v(1), \sigma(0)\}, \cdots, \{v(t-1), v(t), \sigma(t-1)\}, \{v(t), v(t+1), \sigma(t)\}, \cdots)$.

Example 2.1. *(Usage of graphs to characterize switching)*

- *Let Q be a mode-adjacency matrix (as in [6]), that is mode i at time t can be followed by mode j at time $t+1$ only if $Q(i,j) = 1$. A valid graph G could have one node per mode, and an edge (i,j) with a label j whenever $Q(i,j) = 1$.*

- *An unconstrained switching system can of course be represented as a constrained switching system. Any Debruijn graph on N letters represents arbitrary switching.*

A constrained switching system $S(G, \Sigma)$ is said to be *stable* when, for any initial point, for any infinite switching sequence, $\lim_{t \to \infty} |x_t| = 0$. To each path of length t in the graph G, following edges with labels $(\sigma(0), \cdots, \sigma(t-1))$, there is an associated left-product of matrices $(A_{\sigma(t-1)} \cdots A_{\sigma(1)} \cdot A_{\sigma(0)})$. These products are the link between the discrete state and continuous state dynamics : each product corresponds to one discrete-continuous trajectory.

The set of all possible products of any length will be referred to as a *language.*

Definition 2.2 (Language). *A language $\mathcal{L}(G, \Sigma)$, where $G : G(V, E)$ is a labeled directed graph, is the set of all products of matrices in Σ with a sequence of labels that can be mapped on a path in G.*

Also, let $\mathcal{L}_t(G, \Sigma)$ be the set of all admissible products of length t.

The quantity below gives the worst-case growth rate, in term of the norm of x_t, attained after t steps.

Definition 2.3 (Supremum of averaged norm for products of length t)**.**

$$\rho_t(G, \Sigma) = \sup \left(||A||^{1/t} : A \in \mathcal{L}_t(G, \Sigma) \right). \tag{3}$$

Taking the limit $t \to \infty$, we obtain the asymptotic growth rate of the system, that we call the *constrained joint-spectral radius.*

Definition 2.4 (Constrained joint-spectral radius (CJSR))**.** *The constrained joint spectral radius $\rho(G, \Sigma)$ of a finite set of matrices Σ with products in a language $\mathcal{L}(G, \Sigma)$ is defined as*

$$\rho(G, \Sigma) = \lim_{t \to \infty} \rho_t(G, \Sigma). \tag{4}$$

As discussed in the first section, a similar notion has been developed for the unconstrained case (see [5] for a monograph on the topic). Our contribution here lies in the adaptation of this notion to the stability analysis constrained switching systems by considering the underlying switching automaton. In the following, when clear from the context, we will abridge the notation $\rho(G, \Sigma)$ to ρ, and do the same for $\rho_t(G, \Sigma)$.

We can prove that the constrained joint spectral radius is well defined. Note that to do so, we assume that all matrix norms considered are sub-multiplicative, which is the case for induced matrix norms.

Lemma 2.1 (The constrained joint-spectral radius is well defined)**.** *For any bounded set $\Sigma \in \mathbb{R}^{n \times n}$, language-graph $G : G(V, E)$, the function $t \to \rho_t(G, \Sigma)$ converge when $t \to \infty$. Moreover,*

$$\lim_{t \to \infty} \rho_t(G, \Sigma) = \inf\{\rho_t(G, \Sigma)\}. \tag{5}$$

This joint-spectral radius ρ dictates the stability of a switching system $S(G, \Sigma)$.

Theorem 2.1 (Stability)**.** *The switching system $S(G, \Sigma)$ is stable if and only if $\rho(G, \Sigma) < 1$.*

A given system $S(G_1, \Sigma)$ may be equivalent to another system $S(G_2, \Sigma)$. For example the case when G_1 and G_2 are Debruijn graphs on same set of labels (letters) and different number of nodes : any switching sequence in one system can occur in the other.

The following result compares the CJSR of systems sharing the same set of matrices Σ, having some equivalence relationship regarding the switching automaton.

Proposition 2.1. *Consider two graphs G_1 and G_2 such that, for any sequence of label corresponding to a path in G_1, there exists a path in G_2 having the same sequence. Then, for any Σ,*

$$\rho(G_1, \Sigma) \leq \rho(G_2, \Sigma). \tag{6}$$

The proof of Theorem 2.1 implies that whenever $\rho < 1$, all admissible products of matrices have norms going to 0. When $\rho > 1$, there is at least a product whose norm grows to infinity.

In the next section, we will investigate the case where $\rho(G, \Sigma) = 1$. That is, when a CJSR is equal to 1, when do the norms of trajectories remain bounded?

3 Sufficient condition for boundedness when $\rho = 1$

Defectivity defines the behaviour of the system when the CJSR *is equal to 1*.

Definition 3.1 (Non-defectivity). *A language $\mathcal{L} : \mathcal{L}(G, \Sigma)$ is nondefective if there exists $K \in \mathbb{R}$ such that for all t,*

$$\sup\{||A|| : A \in \mathcal{L}_t(G, \Sigma)\} \leq K\rho^t. \tag{7}$$

If a language $\mathcal{L}(G, E)$ is non-defective, then all trajectories of the system $S(G, \Sigma)$ remain bounded. The question of deciding whether a language is non-defective is undecidable (see [5]).

In the following, we will give a sufficient condition for non-defectivity. We prooved that if the constrained switching system has the two properties defined below, then it must be non-defective.

Property 3.1 (Linear connectivity (LC)). *For all $x \in \mathbb{R}^n$, given any two nodes $v_i, v_j \in V$, there exists a path $v_i \to v_j$ and an associated matrix product $A_{(v_i \to v_j)}$ such that $A_{(v_i \to v_j)}x \neq 0$. In that sense, the language* connects *linear subspace from node to node.*

Property 3.2 (No invariant subspaces for a loop (NIS)). *Given any node $v \in N$, for all linear sub-space $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{X} \neq \{0\}$, there exists a cycle $v_i \to v_i$ and $A_{(v_i \to v_i)}$ such that $A_{(v_i \to v_i)}\mathcal{X} \not\subseteq \mathcal{X}$.*

Theorem 3.1. *Consider a language $\mathcal{L}(G, \Sigma)$, with $\rho(G, \Sigma) = 1$, for which both properties (LC) and (NIS) hold. Then, the language is non-defective.*

Once again, note that these conditions mix the structure of the switching automaton and properties on the set Σ, thus taking into account the automaton to get conclusion on the continuous state dynamics.

In the case of arbitrary switching, a well known sufficient condition for non-defectivity is the *irreducibility* (see [5]) of the set Σ. The pairs of conditions above generalizes this irreducibility condition.

4 Nodal norms and approximation

The study of joint-spectral radius, for arbitrary switching, brought negative results that still hold in our case. First, the question $\rho \leq 1$ is undecidable. Also, the problem approximation with relative precision-ϵ, that is the finding of an estimate $\hat{\rho}$ such that $|\hat{\rho} - \rho|/\rho \leq \epsilon$ for a given ϵ, is known to be NP-hard.

In this section, we develop a powerful mathematical construction called *nodal multi-norms*. This construction is inspired from the works [1] and [2], that focused on the arbitrary switching case.

Definition 4.1 (ϵ-nodal multi-norms). *Given a system $S(G, \Sigma)$, a set of ϵ-nodal multi-norms is a set composed one 1 norm per node in $G(V, E)$ having the following property :*

$$\forall e = (v_i, v_j, k) \in E, \; |A_k x|_j \leq (\rho(G, \Sigma) + \epsilon)|x|_i, \tag{8}$$

where $|x|_j, |x|_i$ are the norms associated with node v_i and v_j in the graph that represents the switching automaton of the system.

Proposition 4.1. *Given a system $S(G, \Sigma)$, for all $\epsilon > 0$, there exists a set of ϵ-nodal multi-norms.*

Proposition 4.2. *Given a system $S(G, \Sigma)$, for any $\epsilon < 0$, ϵ-nodal multi-norms cannot exist.*

The two results above already give intuition about how to get *upper bounds* on $\rho(G, \Sigma)$. Assume the system's switching automaton has M states. If one can find a set of norms $|x|_1, \cdots |x|_M$ and $\gamma > 0$, such that for all labeled directed edge $e = \{v_i, v_j, k\} \in E$, $|x|_i \gamma \geq |A_k x|_j$, then γ needs to be an upper bound on $\rho(G, \Sigma)$. A natural approach to approximate the CJSR would be to design a set of nodal multi-norms that would minimize this constant γ, making it closer and closer to the CJSR.

The following result is based on John's Ellipsoid Theorem a powerful theorem on inclusion of ellipsoid in convex sets. It has been used in many application and one closer to ours is presented in [3] for approximations of the joint spectral radius.

This theorem allows to get bounds on the relative precision of an estimation obtained by using *ellipsoidal* multi-norms, i.e. $|x|_i^2 = < Q_i x, x >$, for a given quadratic form Q_i. The notation $Q \succ 0$ stands for Q is positive definite.

Theorem 4.1. *Let $G(V, E)$ represent the switching automaton of a system $S(G, \Sigma)$. The value γ^* such that*

$$\begin{aligned} \gamma^* = \inf_{Q, \gamma \geq 0} & \quad \gamma \\ s.t. & \quad A_k^T Q_j A_k - \gamma Q_i \preceq 0 \quad \forall (v_i, v_j, k) \in E, \\ & \quad Q_i \succ 0; \forall i \in 1 \cdots M. \end{aligned} \tag{9}$$

satisfies the following inequalities :

$$\frac{\sqrt{\gamma^*} - \rho(G, \Sigma)}{\rho(G, \Sigma)} \leq \sqrt{n} - 1. \tag{10}$$

This bound on the relative precision of the estimation can become quite loose as the dimension of the continuous state grows. Nevertheless, the following construction will allow to get arbitrary accurate estimations of the CJSR.

Definition 4.2. *T-lift of a system Given a system $S(G(V, E), \Sigma)$, its T-lift is a system $S(G(V, E^T), \Sigma^T)$ such that*

- *for any path of length T $\{v(1), v(2), \sigma(1)\}, \cdots, \{v(T-1), v(T), \sigma(T)\}$ in G, there is an edge $\{v(1), v(T), (\sigma(1), \cdots, \sigma(T))\} \in E^T$, $(\sigma(1), \cdots, \sigma(T))$ being a concatenation of all labels on the path.*

- *the matrix $A \in \Sigma^T$ associated with a label $(\sigma(1), \cdots, \sigma(T))$ is the product of matrices $A_{\sigma(T)} \cdots A_{\sigma(1)}$, $A_{\sigma(i)} \in \Sigma$.*

Lemma 4.1. *Consider systems $S(G, \Sigma)$ and $S(G(V, E^T), \Sigma^T)$. The following holds true,*

$$\rho(G(V, E^T), \Sigma^T) = \rho(G, \Sigma)^T. \tag{11}$$

Finally, using the above result, we can approximate the CJSR of a given DTLSS arbitrary closely by applying the method of Theorem 4.1 on the T-lift of the system :

Theorem 4.2. *Let $S(G(V, E^T), \Sigma^T)$ be the T-lift of $S(G, \Sigma)$. The value γ^* such that*

$$
\begin{aligned}
\gamma^* = \inf_{Q, \gamma \geq 0} \quad & \gamma \\
s.t. \quad & A_k^T Q_j A_k - \gamma Q_i \prec 0 \quad \forall (v_i, v_j, k) \in E^T, A_k \in \Sigma^T \\
& Q_i \succ 0; \forall i \in 1 \cdots M.
\end{aligned}
\tag{12}
$$

satisfies the following inequalities :

$$
\frac{\sqrt[2T]{\gamma^*} - \rho(G, \Sigma)}{\rho(G, \Sigma)} \leq \sqrt[2T]{n} - 1.
\tag{13}
$$

To conclude this section, given $\epsilon > 0$, a estimator with ϵ-relative precision can be obtained with a T-lift of the system, $T \geq 1/\left(2 \log_n(\epsilon + 1)\right)$.

5 Conclusion

In this work, we considered the switching automaton underlying a discrete-time linear switching system to study its stability. Our results show that it is indeed possible to integrate the automaton in stability analysis. We are able to approximate arbitrary closely the worst-case rate of growth of a system using a construction built upon the switching automaton called multi-norms.

For future work, we are investigating control related issue as the building of stable switching sequence and feedback controllers within our framework. For example, we investigate the relative accuracy of the methods presented in [4], [6].

As a final word, we showed that mixing concepts from automata-theory and convex optimisation (as in Theorems 4.1, 4.2) can lead to fruitful results for stability analysis.

References

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