Left greedy palindromic length

Michelangelo Bucci* and Gwenaël Richomme†

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Abstract

In [A. Frid, S. Puzynina, L.Q. Zamboni, *On palindromic factorization of words*, Adv. in Appl. Math. 50 (2013), 737-748], it was conjectured that any infinite word whose palindromic lengths of factors are bounded is ultimately periodic. We prove this conjecture in a particular case where the palindromic length is replaced with the left greedy palindromic length.

1 Introduction

A fundamental question in Combinatorics on Words is how words can be decomposed into smaller words. For instance, readers can think to some topics presented in the first Lothaire's book [10] like Lyndon words, critical factorization theorem, equations on words, or to the theory of codes [1, 2], or to many related works published since these surveys. As another example, let us mention that in the area of Text Algorithms some decompositions like Crochemore or Lempel-Ziv factorizations play an important role [4, 8]. These factorizations have been extended to infinite words and, for the Fibonacci word, some links have been discovered with the Wen and Wen's decomposition in singular words [3] (see also [7] for a generalization to Sturmian words and see [12, 9] for more on singular words).

In [6], A. Frid, S. Puzynina and L.Q. Zamboni defined the *palindromic length* of a finite word w as the least number of palindromes needed to decompose w. More precisely the palindromic length of w is the least number k such that $w = \pi_1 \cdots \pi_k$ with $\pi_1, \ldots, \pi_k$ palindromes. As in [6] we denote by $|w|_{pal}$ the palindromic length of w. For instance $|abaab|_{pal} = 2$ as $abaab = a.baab$. They conjectured that if the palindromic lengths of factors of an infinite word $\mathbf{w}$ are bounded (we will say that $\mathbf{w}$ has bounded palindromic lengths of factors) then $\mathbf{w}$ is ultimately periodic (that is has the form uv^ω for some words u and v). By a counting argument, they proved that any infinite word with bounded palindromic lengths of factors contains k-powers for arbitrary integers k (it may be emphasized that this condition is true even if we bound the palindromic lengths of prefixes of the infinite word). Moreover, if the word is aperiodic (i.e., not ultimately periodic), each position of the word must be covered by infinitely many runs. In Section 2, we show that the previous conjecture can be restricted to words having infinitely many palindromic prefixes and that, with this restriction, one should expect words to be periodic instead of being ultimately periodic. Moreover we show how to prove the conjecture when palindromic lengths of prefixes are bounded by 2, enumerating the possible forms of palindromic prefixes of an infinite word of such kind.

Let us mention two main difficulties in proving the previous conjecture in the general case. Firstly, when the palindromic lengths of prefixes of an infinite word are not bounded, the function

*michelangelo.bucci@gmail.com

†LIRMM (CNRS, Univ. Montpellier 2) - UMR 5506 - CC 477, 161 rue Ada, 34095, Montpellier Cedex 5 - France and Univ. Paul-Valéry Montpellier 3, Dpt MIAp, Route de Mende, 34199 Montpellier Cedex 5, France

that associates to each integer k the length of the smallest prefix with palindromic length k might grow very slowly. For instance, let us consider the Fibonacci infinite word: as it is the fixed point of the morphism φ defined by $\varphi(a) = ab$ and $\varphi(b) = a$, by [6], the palindromic lengths of prefixes are not bounded. Actually, if $m(k)$ denotes the length of the least nonempty prefix of the Fibonacci word with palindromic length k, one can verifies that $m(1) = 1$, $m(2) = 2$, $m(3) = 9$, $m(4) = 62$, $m(5) = 297$, $m(6) = 1154$, $m(7) = 5473$ and so on. The second problem lies in the fact that a word may have several minimal palindromic factorizations and the palindromic factorizations of a word and of its longest proper prefix are not related. For instance, both words $aabaab$ and $aabaaba$ have palindromic length 2; the first word has two corresponding palindromic factorizations ($aabaa.b$ and $aa.baab$) and it is the longest proper prefix of $aabaaba$, which admits only one unrelated decomposition into two palindromes ($a.abaaba$). To cope with this, in Section 3, we introduce the notion of left greedy palindromic length, which is the number of palindromes in the palindromic decomposition obtained considering iteratively the longest palindromic prefix as the first element. We show that if the left greedy palindromic lengths of prefixes of an infinite word $\mathbf{w}$ having infinitely many palindromic prefixes are bounded then $\mathbf{w}$ is periodic. As it also implies that palindromic lengths of factors of $\mathbf{w}$ are bounded, this proves the Frid, Puzynina and Zamboni's conjecture in a special case.

2 Around the conjecture on palindromic lengths

The result by A. Frid, S. Puzynina and L.Q. Zamboni stating that any infinite word with bounded palindromic lengths of factors contains k-powers for arbitrary integers k is valid also for infinite words with bounded palindromic lengths of prefixes. Although it is an open question whether having bounded palindromic lengths of prefixes implies having bounded palindromic lengths of factors, it seems interesting to consider both properties. For any integer $k \geq 1$, we denote by BPLF(k) (resp. BPLP(k)) the set of all infinite words such that $|u|_{pal} \leq k$ for all their factors u (resp. all their prefixes u). We denote by BPLF the union of sets BPLF(k) and by BPLP the union of sets BPLP(k). Of course, for any integer k, BPLF(k) $\subseteq$ BPLP(k), and BPLF $\subseteq$ BPLP.

It is clear that sets BPLP(1) and BPLF(1) are equal and contain only words on the form a^ω with a a letter. One can observe that, for two different letters a and b and two positive integers i and j, $|ab^ja^ib|_{pal} = 2$ when $i = 1$ or $j = 1$, and, $|ab^ja^ib|_{pal} = 3$ when $i \geq 2$ and $j \geq 2$. Moreover $|ba^iba^jb|_{pal} = 3$ if $i \neq j$, $i \geq 1$ and $j \geq 1$. It follows that words of BPLF(2) $\setminus$ BPLF(1) are the words on the form $a^i(ba^j)^\omega$ with $i \neq 0$ if $j = 0$. Determining words in BPLP(2) is more tedious. Before explaining a way to do it, we restrict our attention to words having infinitely many palindromic prefixes. Next lemma explain the interest of such a restriction. For any word w, we denote by $\tilde{w}$ its mirror image.

Lemma 2.1. *Let* $\mathbf{w}$ *be a word in BPLP.*

1. *All suffixes of* $\mathbf{w}'$ *belong to BPLP.*
2. *There exists a suffix* $\mathbf{w}'$ *of* $\mathbf{w}$ *having infinitely many palindromic prefixes.*
3. *If* $\mathbf{w}$ *have infinitely palindromic prefixes and is ultimately periodic, then* $\mathbf{w}$ *is periodic. Moreover there exists two palindromes (possibly one empty) such that* $\mathbf{w} = (\pi_1\pi_2)^\omega$.

Proof. 1. Let a be a letter and p be a finite word. If $ap = \pi_1\pi_2\cdots\pi_k$ with $\pi_1, \ldots, \pi_k$ palindromes, then $p = \pi_2\ldots\pi_k$ if $\pi_1 = a$, and $p = \pi_1' a\pi_2\ldots\pi_k$ if $\pi_1 = a\pi_1'a$. Thus if $a\mathbf{w}$ belongs to BPLP(k), $\mathbf{w}$ belongs to BPLP($k+1$). In particular if $a\mathbf{w} \in BPLP$ then $\mathbf{w} \in BPLP$. This implies that any suffix of an element of BPLP also belongs to BPLP.

2. Let $I_0 = \{0\}$ and for $k \geq 1$, let $I_k = \{i \mid \exists j \in I_{k-1}, \mathbf{w}[j+1..i] \text{ is a palindrome}\}$. Note that for $k \geq 1$, I_k is the set of all lengths of prefixes of $\mathbf{w}$ that can be decomposed into k palindromes. If all suffixes of $\mathbf{w}$ have only a finite number of palindromic prefixes, then it can be checked quite directly by induction that for all $k \geq 0$, I_k is finite. This contradicts the fact that $\mathbf{w}$ belongs to BPLP. Thus there exists a smallest integer $k \geq 1$ such that I_k is infinite. So there exists $j \in I_{k-1}$, such that $\mathbf{w}[j+1..\infty]$ has infinitely many palindromic prefixes.

3. If $\mathbf{w} = uv^\omega$, there exists a conjugate x of v such that for any integer $k \geq 1$, there is a palindromic prefix of $\mathbf{w}$ ending with x^k. Thus, for all $k \geq 1$, $\tilde{x}^k$ is a prefix of $\mathbf{w}$ and $\mathbf{w} = \tilde{x}^\omega$.

   Now assume $u = \varepsilon$. From what precedes there exists a prefix π_1 of v such that $\mathbf{w}$ has infinitely many prefixes on the form $v^k\pi_1$ with $k \geq 1$: $\mathbf{w} \in (\pi_1\pi_2)^\omega$. Let π_2 such that $v = \pi_1\pi_2$. We let readers verify that π_1 and π_2 are palindromes.

□

From now on, A denotes an alphabet and $\mathcal{P}(A^\omega)$ denotes the set of all infinite words having infinitely many palindromic prefixes. Previous lemma allows to reformulate Frid, Puzynina and Zamboni's conjecture: if $\mathbf{w}$ is an infinite word in BPLF $\cap\, \mathcal{P}(A^\omega)$ then $\mathbf{w}$ is periodic. We suspect that also any infinite word in BPLP $\cap\, \mathcal{P}(A^\omega)$ is periodic. Previous lemma also suggests that any infinite word in $\mathcal{P}(A^\omega) \cap BPLP$ should be uniformly recurrent, but a direct proof of this fact is an open problem.

We now provide a characterization of words in BPLP(2) $\cap\, \mathcal{P}(A^\omega)$.

Lemma 2.2. *An infinite word $\mathbf{w}$ over an alphabet A beginning with the letter a and having infinitely many palindromic prefixes is in BPLP(2) if and only if it has one of the following forms (b is a letter different from a):*

1. $\mathbf{w} = a^\omega$;
2. $\mathbf{w} = (a^iba^j)^\omega$ *for some integers* $i \geq 1$, $j \geq 1$;
3. $\mathbf{w} = (a^ib^j)^\omega$ *for some integers* $i \geq 1$, $j \geq 1$;
4. $\mathbf{w} = ((ab)^ia)^\omega$ *for some integer* $i \geq 2$.

A first step for the proof of the lemma is next result.

Lemma 2.3. *Any infinite word in BPLP(2) $\cap\, \mathcal{P}(A^\omega)$ contains at most two different letters.*

Proof. Assume that $\mathbf{w}$ contains at least three letters. Then it has a prefix on the form pc with p containing exactly two different letters a and b, and with c a letter different from a and b. Inequality $|pc|_{pal} \leq 2$ implies $|p|_{pal} = 1$. Let $pcx\tilde{p}$ be the shortest palindromic prefix of $\mathbf{w}$ having pc as a prefix (cx ends with c). Assume a is the first letter of p. There exists an integer $n \geq 1$ and a word y such that $p = a^nby$. Let $u = pcx\tilde{y}$. We let readers verify that $|u|_{pal} = 3$. □

In order to prove Lemma 2.2, we consider the possible palindromic prefixes of $\mathbf{w}$. By $\text{next}(u)$ we denote the set of all palindromes over $\{a, b\}$ having u as a proper prefix, such that all proper palindromic prefixes are prefixes of u and such that $|p|_{pal} \leq 2$ for all their prefixes p. For instance $\text{next}(a^i) = \{a^{i+1}\} \cup \text{next}(a^ib)$

Lemma 2.4. *For any integer $i \geq 1$,*

1. $\text{next}(a^ib) = \{a^ib^ja^i \mid j \geq 1\} \cup \{a^i(ba^j)^kba^i \mid k \geq 1, 1 \leq j < i\}$;

2. $\text{next}(a^i(ba^j)^k ba^i) = \emptyset$ *when* $1 \le j < i$, $k \ge 1$;
3. $\text{next}(a^i(b^j a^i)^k a) = \emptyset$ *when* $j \ge 2$ *and* $k \ge 1$;
4. $\text{next}(a^i(b^j a^i)^k b) = \{a^i(b^j a^i)^{k+1}\}$ *when* $j \ge 1$ *and* $k \ge 1$;
5. $\text{next}(a^i(ba^i)^k a) = \emptyset$ *when* $i \ge 2$ *and* $k \ge 2$;
6. $\text{next}(a^i ba^{i+1}) = \{a^i ba^{i+j} ba^i \mid j \ge 1\}$.
7. $\text{next}(a^i(ba^{i+j})^k ba^i) = \{a^i(ba^{i+j})^{k+1} ba^i\}$ *when* $j \ge 1$, $k \ge 1$;
8. $\text{next}(a(ba)^k a) = \{(a(ba)^k)^2\}$ *when* $k \ge 2$.
9. $\text{next}((a(ba)^k)^\alpha) = \{(a(ba)^k)^{\alpha+1}\}$ *when* $k \ge 2$ *and* $\alpha \ge 2$

The combinatorial proof of each item of the previous lemma, which is rather tedious, but not difficult, is omitted for lack of space.

Proof of Lemma 2.2. The if part is straightforward. Let $\mathbf{w}$ be an infinite word in BPLP(2) $\cap$ $\mathcal{P}(A^\omega)$ beginning with the letter a. By Lemma 2.3, $\mathbf{w}$ contains at most two different letters. Let b be the second possible letter. For some integer $i \ge 1$, $\mathbf{w}$ begins with $a^i b$. By Items 1 and 2 of Lemma 2.4, for some integer $j \ge 1$, $\mathbf{w}$ begins with $a^i b^j a^i$. Assume $\mathbf{w} \neq (a^i b^j)^\omega$. Let $k \ge 1$ be the maximal integer such that $a^i(b^j a^i)^k$ is a prefix of $\mathbf{w}$. By Item 4 of Lemma 2.4, $\mathbf{w}$ begins with $a^i(b^j a^i)^k a$ for some $k \ge 1$. By Item 3 of Lemma 2.4, $j = 1$. Assume $k = 1$. By Item 6 of Lemma 2.4 there exists an integer $j' \ge 1$ such that $\mathbf{w}$ begins with $a^i b^{i+j'} ba^i$. Thus by Item 7 of Lemma 2.4, $\mathbf{w} = (a^i ba^{j'})^\omega$. Assume from now on that $k \ge 2$. By Item 5 of Lemma 2.4, $i = 1$ and by Items 8 and 9 of Lemma 2.4, $\mathbf{w} = (a(ba)^k)^\omega$. $\square$

Maybe the main interest of previous proof is the idea of studying the links between successive palindromic prefixes of the considered infinite words. Determining BPLF(3) seems much more difficult, as a simple case analysis seems unfeasible for the large number of cases and the fact that words, such as $(abac)^\omega$, that contain three different letters must be considered.

3 Left greedy palindromic length

As explained in the introduction, we introduce a new measure of complexity for finite words, the left greedy palindromic length, and show that if a word $\mathbf{w}$ in $\mathcal{P}(A^\omega)$ is such that the left greedy palindromic lengths of its prefixes are bounded, then $\mathbf{w}$ is periodic.

The *left greedy palindromic length* of a word w is defined inductively by: $|w|_{LgPal} = 0$ when w is the empty word, $|w|_{LgPal} = 1 + |u|$ if $w = \pi u$ with π the longest palindromic prefix of w. For instance, $|abaa|_{LgPal} = 2$ and $|abaab|_{LgPal} = 3$. Similarly, we define the *right greedy palindromic length* $|w|_{RgPal}$ considering at each step the longest palindromic suffix: $|abaa|_{RgPal} = 3$ and $|abaab|_{RgPal} = 2$.

Property 3.1. *For any word u, $|u|_{pal} \le \min(|u|_{LgPal}, |u|_{RgPal})$.*

As a consequence, if for an infinite word $\mathbf{w}$ there exists an integer K such that $|p|_{LgPal} \le K$ for any prefix p of $\mathbf{w}$, then also $|p|_{pal} \le K$. We have the stronger property:

Property 3.2. *If for an infinite word $\mathbf{w}$ there exists an integer K such that $|p|_{LgPal} \le K$ for any prefix p of $\mathbf{w}$, then also $|u|_{pal} \le 2K$ for any factor u of $\mathbf{w}$.*

Proof. Let u be a factor of $\mathbf{w}$ and let p be such that pu is a prefix of $\mathbf{w}$. By hypothesis, there exist $\pi_1, \ldots, \pi_k$ palindromes such that $1 \leq k \leq K$ and $pu = \pi_1 \cdots \pi_k$. Let x, y be the words and i be the integer such that $\pi_i = xy$ and $u = y\pi_{i+1} \cdots \pi_k$. The word $\tilde{y}$ is a prefix of the palindrome π_i. Hence $\pi_1 \cdots \pi_{i-1}\tilde{y}$ is a prefix of $\mathbf{w}$. By hypothesis, there exist palindromes $\pi'_1, \ldots, \pi'_{k'}$ such that $k \leq K$ and $\pi_1 \cdots \pi_{i-1}\tilde{y} = \pi'_1 \cdots \pi'_{k'}$. As, by definition, palindromes π_j and π'_j have to be chosen as the longest palindromic prefixes of respectively $\pi_j \cdots \pi_k$ and $\pi'_j \cdots \pi'_{k'}$, it follows that $\pi_1 = \pi'_1, \ldots, \pi_{i-1} = \pi'_{i-1}$ and $\tilde{y} = \pi'_i \cdots \pi'_{k'}$. Hence $u = \pi'_{k'} \cdots \pi'_i \pi_{i+1} \cdots \pi_k$ is the product of at most $2K$ palindromes. □

One can observe that the words in $\mathcal{P}(A^\omega)$ such that $|p|_{LgPal} \leq 2$ for any prefix p are necessarily words of the form a^ω or $(ab^k)^\omega$ (a and b are letters). Indeed for any integer $k, n \geq 1$, $|a^i b^k a|_{LgPal} = 3$ if $i \geq 2$, $|(ab^k)^n a^i b|_{LgPal} = 3$ if $i \geq 2$ and $|(ab^k)^n ab^i a|_{LgPal} = 3$ if $i \neq k$.

Now we state our main result.

Theorem 3.3. *If the left greedy palindromic lengths of prefixes of a word $\mathbf{w} \in \mathcal{P}(A^\omega)$ are bounded, then $\mathbf{w}$ is periodic.*

As a consequence of Lemma 2.1, we have as a corollary that if the left greedy palindromic lengths of prefixes of a word $\mathbf{w}$ are bounded, then $\mathbf{w}$ is ultimately periodic.

The proof of Theorem 3.3 is an adaptation of the proof of next result which is nothing else than Theorem 3.3 in the special case of words containing infinitely many different letters.

Lemma 3.4. *For any infinite word $\mathbf{w}$ over an* infinite *alphabet having infinitely many palindromic prefixes, the set $\{|p|_{LgPal} \mid p$ prefix of $\mathbf{w}\}$ is unbounded.*

Proof. Let $(\pi_n)_{n\geq 0}$ be the sequence of palindromic prefixes of $\mathbf{w}$ (all palindromic prefixes of $\mathbf{w}$ occur in the sequence and $(|\pi_n|)_{n\geq 0}$ is (strictly) increasing). Let n be an integer such that a letter α occurs in π_{n+1} but not in π_n. Then $\pi_{n+1} = \pi_n \pi \pi_n$. Let p be a proper prefix of π_n:

$$|\pi_n \pi p|_{LgPal} = 2 + |p|_{LgPal} \tag{1}$$

Indeed, one can observe first that by definition of n and palindromes $(\pi_i)_{i\geq 0}$, π_n is the longest palindromic prefix of $\pi_n \pi p$. Now we have to verify that π is the longest palindromic prefix of πp. Let us write $\pi = x\alpha y$ with $|x|_\alpha = 0$ and assume that πz is a palindromic prefix of πp. As $\pi_n = ps = zts$ for words s and t, and as π_n and π_{n+1} are palindromes, we deduce that $\alpha\tilde{x}$ is a suffix of z and $\tilde{s}\tilde{t}x\alpha$ is a prefix of π_{n+1}. As $|\tilde{s}\tilde{t}x|_\alpha = 0$ (as $|\pi_n|_\alpha = 0$), necessarily $|t| = |p|$.

As $\mathbf{w}$ contains infinitely many letters, Equation (1) implies the lemma. □

In the adaptation of the previous proof to Theorem 3.3, the infinite alphabet is replaced with infinitely many powers of a given word. The existence of such powers is guaranteed by an inductive hypothesis. More precisely, given an infinite word $\mathbf{w}$, let $\mathrm{MaxLgPalPref}(\mathbf{w})$ be the supremum of the set $\{|p|_{LgPal} \mid p$ prefix of $\mathbf{w}\}$. Our proof of Theorem 3.3 acts by induction on $\mathrm{MaxLgPalPref}(\mathbf{w})$. The only infinite words such that all their nonempty prefixes are of left greedy palindromic length 1 are words on the form a^ω. Assume that K is an integer such that, for any infinite word $\mathbf{x}$ in $\mathcal{P}(A^\omega)$, $\mathrm{MaxLgPalPref}(\mathbf{x}) \leq K$ implies that $\mathbf{x}$ is periodic. Let $\mathbf{w}$ be such that $\mathrm{MaxLgPalPref}(\mathbf{w}) = K + 1$.

Lemma 3.5. *There exist nonempty palindromes u and v, a length increasing sequence $(\pi_i)_{i\geq 0}$ of palindromes and a strictly increasing sequence of integers $(\ell_i)_{i\geq 1}$ such that $\pi_{i-1}^{-1}\pi_i$ begins with $(uv)^{\ell_i}u$ for all $i \geq 1$.*

Proof. Let $(\pi_i)_{i\geq 1}$ be the length increasing sequence of palindromic prefixes of $\mathbf{w}$. Let i be an integer and let p be any word such that $\pi_i p$ is a proper prefix of π_{i+1}. By definition of the sequence $(\pi_j)_{j\geq 1}$, we have $|\pi_i p|_{LgPal} = 1 + |p|_{LgPal} \leq K + 1$. So for all proper prefixes p of $\pi_i^{-1}\pi_{i+1}$ we have$|p|_{LgPal} \leq K$. One can prove that the sequence $(|\pi_{i+1}| - |\pi_i|)_{i\geq 1}$ is not decreasing and unbounded. Thus by König's lemma (see [11, Prop. 1.2.3]), there exists an infinite word $\mathbf{w}'$ such that each of its prefixes p is a prefix of $\pi_j^{-1}\pi_{j+1}$ for some $j \geq 1$. From what precedes $\mathrm{MaxLgPalPref}(\mathbf{w}') = K$ and, by hypothesis, $\mathbf{w}'$ is periodic.

As $\mathbf{w} \in \mathcal{P}(A^\omega)$, also $\mathbf{w}' \in \mathcal{P}(A^\omega)$. Thus by Item 3 of Lemma 2.1, $\mathbf{w}' = (uv)^\omega$ for two palindromes u and v. There exist two sequences of integers $(\ell_i)_{i\geq 1}$ and $(j_i)_{i\geq 0}$ such that $(uv)^{\ell_i}u$ is a prefix of $\pi_{j_i}^{-1}\pi_{j_i+1}$ which itself is a prefix of $\pi_{j_i}^{-1}\pi_{j_{i+1}}$. The sequence $(\ell_i)_{i\geq 1}$ can be chosen strictly increasing. Replacing $(\pi_i)_{i\geq 0}$ by the sequence $(\pi_{j_i})_{i\geq 0}$ ends the proof. ☐

When $\mathbf{w}$ is not periodic, the previous fact implies that the set of factors of the form $(uv)^\ell u$ not preceded by uv and not followed by vu is infinite. This set plays the role of an infinite alphabet, and, not trivially, the proof of Lemma 3.4 can be adapted to state a contradiction.

4 Conclusion

Property 3.1 is that, for any word u, $|u|_{pal} \leq \min(|u|_{LgPal}, |u|_{RgPal})$. The next example shows that the value of $\min(|u|_{LgPal}, |u|_{RgPal}) - |u|_{pal}$ can be arbitrarily large.

Let f_n be the nth Fibonacci word ($f_1 = a$, $f_2 = ab$, $f_n = f_n f_{n-1}$ for $n \geq 3$). Z.-X. Wen and Z.-Y. Wen [12] introduced the notion of singular words for the Fibonacci word. Let $(\pi_n)_{n\geq 2}$ be the sequence of these singular words and let $\alpha_n\beta_n$ denotes the length 2 suffix of f_n (with α_n, β_n letters): $\pi_n = \alpha f_n \beta^{-1}$. Moreover (see [5]), for $n \geq 2$, $f_n = B_n\alpha_n\beta_n$ where B_n is the nth nonempty bispecial factor. All singular words are primitive palindromes. Moreover π_n is the longest palindromic suffix of $B_n\alpha_n$. Thus one can verify that $|\alpha_n B_n|_{LgPal} = |B_n\alpha_n|_{RgPal} = n$ while $|\alpha_n B_n|_{pal} = |B_n\alpha_n|_{pal} = 2$. Taking two fresh letters c and d, letting $u_n = \alpha_n B_n cd B_n \alpha_n$, we obtain $|u_n|_{LgPal} = n + 4 = |u_n|_{RgPal}$ while $|u_n|_{pal} = 6$.

By Lemma 2.1 one can see that for a periodic word in BPLP, the value of $\min(|u|_{LgPal}, |u|_{RgPal}) - |u|_{pal}$ taken over factors u of $\mathbf{w}$ cannot be arbitrarily large. Finally to summarize this paper, observe that that the Frid, Puzynina and Zamboni's conjecture could be stated as follows. For an infinite word $\mathbf{w}$ having infinitely many palindromic prefixes, the following assertions are equivalent:

1. $\mathbf{w}$ has bounded palindromic lengths of factors;
2. $\mathbf{w}$ has bounded palindromic lengths of prefixes;
3. $\mathbf{w}$ has bounded left greedy palindromic lengths of factors;
4. $\mathbf{w}$ has bounded left greedy palindromic lengths of prefixes;
5. $\mathbf{w}$ is periodic;
6. $\mathbf{w} = (uv)^\omega$ for two palindromes u and v.

Equivalence between assertions 3 to 6 are proved in this paper. Indeed Equivalence between assertions 5 and 6 is provided by item 3 of Lemma 2.1. Clearly assertion 6 implies assertions 3 which implies assertion 4. Finally Theorem 3.3 states that assertion 4 implies assertion 5.

Clearly assertion 6 implies assertions 1 which implies assertion 2. That assertion 1 or assertion 2 implies assertion 6 stays an open problem.

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