On the synchronizing probability function and the triple rendezvous time as approaches to Černý's conjecture

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Abstract

We push further a recently proposed method for studying synchronizing automata and Černý's conjecture, namely, the *synchronizing probability function*. In this method, the synchronizing phenomenon is reinterpreted as a Two-Player game, in which the optimal strategies of the players can be obtained through a Linear Program. Our analysis will mainly focus on the concept of the *triple rendezvous time*, the shortest length of a word mapping three states onto a single one. It represents an intermediate step in the synchronizing process, and is a good characterization of its overall length.

Our contribution is twofold. First, using the synchronizing probability function and properties of linear programming, we provide a new upper bound on the triple rendezvous time. Second, we disprove a conjecture on the synchronizing probability function by exhibiting a family of counterexamples. We discuss the game theoretic approach and possible further work in the light of our results.

1 Synchronizing automata and Černý's conjecture

An automaton is called *synchronizing* if there exists a sequence of letters which maps all the states onto a single one (see next subsection for rigorous definitions). Figure 1 shows an example of such an automaton.

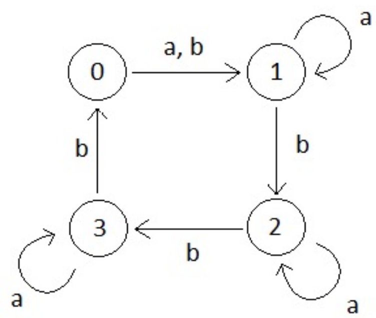

Figure 1: A synchronizing automaton on four states with two letters, a and b. The sequence of letters *abbbabbba* leads to state 1, whatever the initial state is.

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Synchronizing automata have been the source of intense and exciting research in the past 50 years. The interest for the subject appeared in computers and relay control systems in the 60s. The aim was to restore the control over these devices without knowing their current state (see [13], [10]). In the 80s and 90s, the use of synchronizing automata spread through applications in robotics and industry. More recently, synchronizing automata have found biological applications. The survey [19] gives a more detailed overview of synchronizing automata applications.

In this paper, we represent automata with matrices. In this representation, a set of states $S \subset Q$ of an automaton (with n possible states) is represented by its characteristic vector, i.e. a $1 \times n$ vector $x \subset \{0,1\}^n$ for which $x_i = 1$ if state $i \in S$, and 0 otherwise. A state is represented by such a binary vector with a single entry equal to one. The letters of the automaton will be represented as $n \times n$ matrices, such that applying the letter a represented by matrix A to an automaton with characteristic vector x is equivalent to computing the product xA. More precisely:

A *(deterministic, finite state, complete) automaton* (DFA) is a set of m column-stochastic matrices $\Sigma \subset \{0,1\}^{n\times n}$ (where m, n are respectively the number of letters in the alphabet, and the number of states of the automaton). That is, the matrices in Σ have binary entries, and they satisfy $Ae^T = e$, where e is the $1 \times n$ all-ones vector. We write Σ^t for the set of matrices which are products of length t of matrices taken in Σ. We refer to these matrices as *words* of length t.

Definition 1. *An automaton $\Sigma \subset \{0,1\}^{n\times n}$ is* synchronizing *if there is a finite product $A = A_{c_1} \dots A_{c_W} : A_{c_i} \in \Sigma$ and an index $0 < i \leq n$ which satisfy*

$$A = e_i^T e,$$

where e_i is the ith standard basis vector $(1 \times n)$.
In this case, the sequence of letters $c_1 \dots c_W$ is said to be a synchronizing word.

Jan Černý stated his conjecture on DFA in 1964 [6] [7]. Although it is very simple in its formulation, it has not been proven since then. The conjecture is the following:

Conjecture 1. ***Černý's conjecture, 1964 [7]*** *Let $\Sigma \subset \{0,1\}^{n\times n}$ be a synchronizing automaton. Then, there is a synchronizing word of length at most $(n-1)^2$.*

In [6], Černý proposes an infinite family of automata attaining this bound, for any number of states. The automaton represented in Figure 1 is the representative of this family on four states. Indeed, its smallest synchronizing word is *abbbabbba*, which is composed of nine letters. The examples of synchronizing automata attaining this bound, or even getting close to it for a large number of states, are very infrequent (some families of such automata are developed in [1]).

Since its formulation, Conjecture 1 has been the subject of intense research. Up to now, the best upper bound on the length of a minimal synchronizing word for an automaton of size n is even not quadratic, but is equal to $(n^3 - n)/6$ [14] [15]. This bound in the general case has been holding for more than 30 years. Conjecture 1 has been proven to hold for subfamilies of automata [2,3,5,6,8,9,12,18]. Very recently, several attempts to introduce a probabilistic point of view to this problem have appeared in the literature (see [17] for a recent presentation of the main ideas). A state of the art overview is presented in [19].

Our work is based on the *synchronizing probability function*, introduced in 2011 [11]. This tool allows to reformulate the synchronizing property in terms of game theory. It is promising in that it offers a connection with an a priori unrelated concept, which relies on a strong theoretical basis (see [4], [16]).

In the next section, we will introduce the concept of *triple rendezvous time*, and obtain an upper bound on this value, based on the synchronizing probability function. In Section 3, we will describe a particular family of automata which, amongst other interesting properties, refutes a recent conjecture on the synchronizing probability function. We will not redefine here the concept of synchronizing probability function, which can be found in [11], nor give the full proofs of the results, which will appear in later publications.

2 Upper bound on the triple rendezvous time

In what follows, the *weight* of a vector is the number of its non-zero elements. As defined in [11], $A(t)$ is the bloc-row matrix containing all the matrix representations of words of length at most t of the automaton considered.

Definition 2. *Let us consider a synchronizing automaton $\sum$. We define the* triple rendezvous time *$T_{3,\sum}$ as the smallest integer such that $A(T_{3,\sum})$ contains a column of weight superior or equal to 3.*

In other words, it is equal to the length of the smallest word for which one of the states has a pre-image of cardinality three. In the following, we will use T_3 for $T_{3,\sum}$ when there is no ambiguity on the automaton.

Although the concept of *triple rendezvous time* is a very natural concept, we are not aware of any attempt to bound its value for synchronizing automata.

We will obtain an upper bound on the triple rendezvous time by using the synchronizing probability function. See [11] for a more detailed overview of this function.

Our motivation for studying T_3 is that there are empirical evidences that its value is correlated with the length of a shorter synchronizing word. Indeed, for the known automata achieving the bound of Conjecture (1), T_3 is around n, and the SPF is growing close to linearly. These considerations led to the following conjecture, which states that, although the SPF can deviate from a linear growth, at time equal to multiples of $(n+1)$, it has to catch up with the slope corresponding to Conjecture 1:

Conjecture 2. *in [11] **(Conjecture 2)** For any synchronizing automaton A and for any $j \geq 1, j \leq n-1$,*

$$k(1+(j-1)(n+1)) \geq j/(n-1).$$

This conjecture would imply Černý's conjecture [11]. It also would imply that the triple rendezvous time for an automaton on n nodes is lower or equal to $n+2$ (see [11], section 4). We will see later that it is not the case.

We now present an upper bound on T_3 (see GandALF 2014 proceedings for the proofs). First, it is not difficult to see that in a synchronizing automaton $\sum$ with n states, $T_{3,\sum} \leq \frac{n(n-1)}{2} + 1$. Indeed, there are $\frac{n(n-1)}{2}$ possible pairs of states, and therefore as $A(t)$ grows and has to evolve at each step, we obtain our result. However, it is possible to improve this bound:

Theorem 1. *In a synchronizing automaton with* n *states,* $T_3 \leq \frac{(n)(n+4)}{4} - \frac{\textit{Parity } n}{4}$*, with Parity* $n =$ 0 *if* n *is even, and* 1 *if* n *is odd.*

Proof. (sketch) We have the following lemma:

Lemma 1. *if* $t < T_3$*, then* $k(t)$ *can only take the values* $2/(n+s)$, $0 \leq s \leq n-1$*, and this value cannot be optimal at more than* $\lfloor (n-s)/2 \rfloor + 1$ *steps.*

This allows us to derive a better bound on T_3 by counting all the possible steps before T_3:

$$\sum_{s=0}^{n-1} (\lfloor (n-s)/2 \rfloor + 1) = \sum_{s=1}^{n} (\lfloor s/2 \rfloor + 1) = \frac{1}{2}\left(\sum_{l=0}^{n-1} l\right) - \frac{1}{2}\lceil \frac{n}{2} \rceil + n = \frac{(n)(n+4)}{4} - \frac{\text{Parity n}}{4}$$

□

3 Counterexample to a conjecture on the synchronizing probability function

In this section, we present an infinite family of automata which are counterexamples to Conjecture 2. It also provides us with a lower bound on the maximum value of the triple rendezvous time of an automaton of size n for every odd number $n \geq 9$.

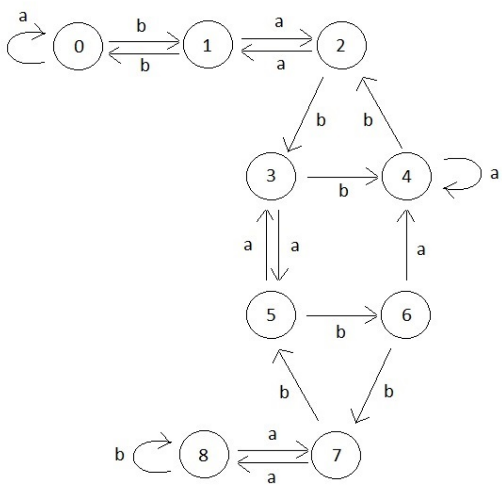

Figure 2: Automaton on 9 nodes with $k(11) = 2/9$.

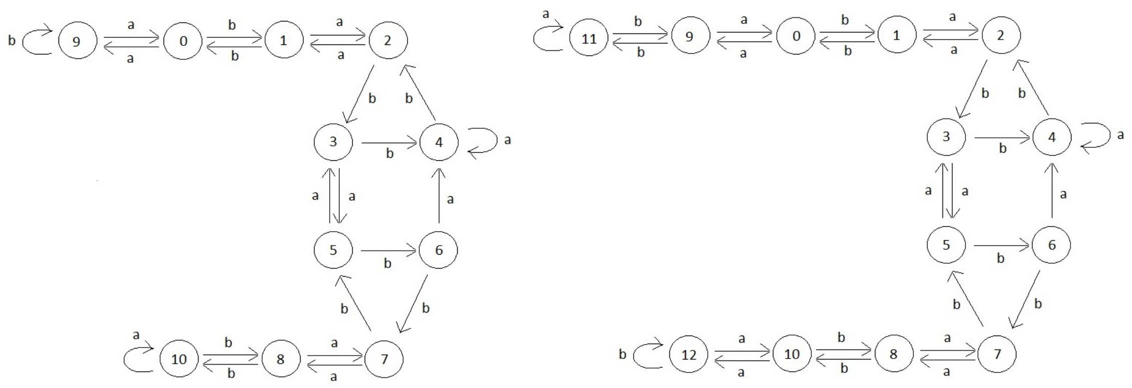

Figure 3: Representation of the automata of this family on 11 and 13 states

The automaton on nine nodes and two letters represented in Figure 2 is the first representative of the family. Indeed, its synchronizing probability function with $t = 11$ is $k(11) = 2/9$. However, for $j = 2$ and $n = 9$, Conjecture 2 states that $k(11) \geq 2/8$, and is therefore false. This automaton also has the particularity that its triple rendezvous time equals 12, which is the number of nodes plus 3. Indeed, it can be verified that he matrix $A(11)$ contains only columns of weight two. In addition, for the three initial states 0, 4 and 6 of the automaton, the automaton ends at state 4 after application of the word *abbabbababba*, which is twelve letters long.

We can now extend this automaton on 9 nodes to an infinite family of automata with an odd number of nodes. The construction of the automaton on 11 nodes is the following: taking the same initial structure, we first delete the self loop with label a on node 0, and add node 9 with a double connection with label a to node 0. Second, we add a self loop of label b on node 9. Third, we delete the self loop with label b on node 8, and add node 10 with a double connection with label b to node 8. Fourth, we add a self loop with label a on node 10. This gives us the automaton on 11 nodes shown in Figure 3. We can do the same process to get to higher steps: at each step, we delete the self loops on the nodes added in the previous step. Then we add new nodes connected to them with a double connection labelled with the letter of the deleted loops. Then we add a self loop labelled with the other letter, as shown in Figure 3.

All the automata of this family are such that $T_3 = n + 3$, and $k(n + 2) = 2/n$.

4 Conclusion

In this work, we pushed further the study of the synchronizing probability function as a tool to represent the synchronization of an automaton. Our results are twofold, and somewhat antagonistic: on the one hand, we managed to prove a non trivial upper bound on the triple rendezvous time thanks to results on the synchronizing probability function. It shows that this tool can effectively help in understanding synchronizing automata. On the other hand, we refuted Conjecture 2, by providing an infinite family of automata for which $T_3 = n + 3$ (with n the size of the automaton). Conjecture 2 was stated as a tentative roadmap toward a proof of Cerny's conjecture with the help of the synchronizing probability function, and in that sense our conterexample is a negative result towards that direction.

A natural continuation to this research would be to search for the smallest value at which a column of weight larger than three appears. More precisely to search for bounds on the "quadruple rendezvous time" and higher steps, since Černý's conjecture is about the n-uple rendezvous time. Another research question is how to narrow the gap between $n+3$ and $n^2/4$ for the triple rendezvous time, which is in our view interesting per se.

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