Recognizable Series on Hypergraphs

Raphaël Bailly, Guillaume Rabusseau, and François Denis*

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Abstract

We introduce the notion of *Hypergraph Weighted Model* (HWM) that generically associates a tensor network to a hypergraph and then computes a value by tensor contractions directed by its hyperedges. A series r defined on a hypergraph family is said to be recognizable if there exists a HWM that computes it. This model generalizes the notion of rational series on strings and trees. We prove some closure properties and study at which conditions, finite support series are recognizable.

1 Introduction

Real-valued functions whose domains are composed of syntactical structures, such as strings, trees or graphs, are widely used in computer science. One way to handle them is by means of rational series that use automata devices to jointly analyse the structure and compute its image. Rational series have been defined for strings and trees, but their extension to graphs is challenging.

On the other hand, rational series have equivalent algebraic characterisations by means of linear (or multi-linear) representations [3] [4]. We show in this paper that this last formalism can be naturally extended to graphs or hypergraphs by associating tensors to the vertices of the graph.

More precisely, we define the notion of *Hypergraph Weighted Model* (HWM), a computational model that generically associates a tensor network [9] to a hypergraph and that computes a value by successive generalized tensor contractions directed by its hyperedges. We say that a series r defined on a hypergraph family is recognizable (by HWM) if there exists a HWM M that computes it: we then denote r by r_M. We first show that recognizable series defined on strings or trees exactly recover the notion of rational series, while they can be defined on much more general families. We prove two closure properties: if r and s are two recognizable series defined on a family of connected hypergraphs, then $r + s$ and $r \cdot s$, respectively defined by $(r + s)(G) = r(G) + s(G)$ and $(r \cdot s)(G) = r(G)s(G)$ (the Hadamard product) are recognizable.

Rational series on strings and trees include polynomes, i.e. finite support series. This is not always the case for recognizable series. For example, we show that finite support series are not recognizable on the family of circular graphs (or strings). The main reason is that if a recognizable series is not null on some hypergraph G, it must be also different from zero on *tilings* of G, i.e. connected graphs made of copies of G. We show that if a graph family is tiling-free, then recognizable series contain finite support series. Strings and trees, as any family of rooted hypergraphs, are tiling-free.

We recall notions on tensors and hypergraphs in Section 2, we introduce the Hypergraph Weighted Model and we study some closure properties in section 3, we introduce the notion of tilings and we study the recognizability of finite support series in Section 4, and we then propose a short conclusion.

All the proofs have been omitted in this extended abstract but can be found in [1].

*rbailly@lsi.upc.edu, guillaume.rabusseau@lif.univ-mrs.fr, francois.denis@lif.univ-mrs.fr

2 Preliminaries

2.1 Rational Series on Strings and Trees

We refer to [3, 6, 4] for notions about rational series on strings and trees.

Let Σ be a finite *alphabet*, and Σ^* be the set of strings on Σ. A *series* on Σ^* is a mapping $r : \Sigma^* \to \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A series r is *recognizable* (or *rational*) if there exists a tuple (V, ι, μ, τ) where $V = \mathbb{K}^d$ for some integer $d \geq 1$, $\iota, \tau \in V$ and μ maps each symbol $x \in \Sigma$ to a square matrix $M_x \in \mathbb{K}^{d\times d}$, such that for any $u_1 \dots u_n \in \Sigma^*$, $r(u_1 \dots u_n) = \iota^\top M_{u_1} \dots M_{u_n}\tau$.

A *ranked* alphabet $\mathcal{F}$ is a tuple $(\Sigma, \sharp)$ where Σ is a finite alphabet and where $\sharp$ maps each symbol x of Σ to an integer $\sharp x$ called its *arity*; for any $k \in \mathbb{N}$, let us denote $\mathcal{F}_k = \sharp^{-1}(\{k\})$. A ranked alphabet is *positive* if $\sharp$ takes its values in $\mathbb{N}_+$.

The set of trees over a ranked alphabet $\mathcal{F}$ is denoted by $T(\mathcal{F})$. A *tree series* on $T(\mathcal{F})$ is a mapping $r : T(\mathcal{F}) \to \mathbb{K}$. A series r is *recognizable* (or *rational*) if there exists a triple (V, μ, λ), where $V = \mathbb{K}^d$ for some integer $d \geq 1$, μ maps each $f \in \mathcal{F}_p$ to a p-multilinear mapping $\mu(f) \in \mathcal{L}(V^p; V)$ for each $p \geq 0$ and $\lambda \in V$, such that $r(t) = \lambda^\top \mu(t)$ for all t in $T(\mathcal{F})$, where $\mu(t) \in V$ is inductively defined by $\mu(f(t_1, \dots, t_p)) = \mu(f)(\mu(t_1), \dots, \mu(t_p))$.

2.2 Tensors

Let $d \geq 1$ be an integer, $V = \mathbb{K}^d$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ be the canonical basis of V. A tensor $\boldsymbol{\mathcal{T}} \in \bigotimes^k V = V \otimes \cdots \otimes V$ (k times) can uniquely be expressed as a linear combination $\boldsymbol{\mathcal{T}} = \sum_{i_1,\dots,i_k \in [d]} \boldsymbol{\mathcal{T}}_{i_1 \dots i_k} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}$ (where $[d] = \{1, \cdots, d\}$) of *pure tensors* $\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}$ which form a basis of $\bigotimes^k V$ [8]. Hence, the tensor $\boldsymbol{\mathcal{T}}$ can be represented as the multi-array $(\boldsymbol{\mathcal{T}}_{i_1 \dots i_d})$.

Definition 1. *The* tensor product *of $\boldsymbol{\mathcal{T}} \in \bigotimes^p V$ and $\boldsymbol{\mathcal{U}} \in \bigotimes^q V$ is the tensor $\boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{U}} \in \bigotimes^{p+q} V$ defined by $(\boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{U}})_{i_1 \cdots i_p j_1 \cdots j_q} = \boldsymbol{\mathcal{T}}_{i_1 \cdots i_p} \boldsymbol{\mathcal{U}}_{j_1 \cdots j_q}$. For any $\mathbf{v} \in \mathbb{K}^d$, $\mathbf{v}^{\otimes^k} = \mathbf{v} \otimes \cdots \otimes \mathbf{v} = \sum_{i_1,\dots,i_k \in [d]} v_{i_1} \dots v_{i_k} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}$ is the k-th tensor power of $\mathbf{v}$.*

Definition 2. *For any $\alpha \in V$, $\boldsymbol{\mathcal{T}} \in \bigotimes^k V$ and $j \in [k]$, let us define $\alpha \cdot \boldsymbol{\mathcal{T}} \in \mathbb{K}$ and $\alpha \cdot_j \boldsymbol{\mathcal{T}} \in \bigotimes^{k-1} V$ by $\alpha \cdot (\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}) = \alpha^\top \mathbf{e}_{i_1} \times \cdots \times \alpha^\top \mathbf{e}_{i_k}$ and $\alpha \cdot_j (\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}) = \alpha^\top \mathbf{e}_{i_j} (\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_{j-1}} \otimes \mathbf{e}_{i_{j+1}} \otimes \cdots \otimes \mathbf{e}_{i_k})$, and by extending these relations by linearity.*

Let $\odot : V \times V \to V$ be an associative and symmetric bilinear mapping: $\forall u, v, w \in V, u \odot v = v \odot u$ and $u \odot (v \odot w) = (u \odot v) \odot w$. The mapping $\odot$ is called a *product*.

Remark 1. *Let $\alpha = \mathbf{1} = (1, \dots, 1)^\top$ and let $\odot_{id}$ be defined by $\mathbf{e_i} \odot_{id} \mathbf{e_j} = \delta_{ij} \mathbf{e_i}$, where δ is the Kronecker symbol: $\odot_{id}$ is called the* identity product.

Let $m < n \leq k$ be integers. Using our notations, the usual (m,n)-contraction operator $\mathcal{C}_{m,n} : \bigotimes^k V \to \bigotimes^{k-2} V$ can be defined by $\mathcal{C}_{m,n}(\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}) = \alpha \cdot_m (\mathbf{e}_{i_1} \otimes \cdots \mathbf{e}_{i_{m-1}} \otimes (\mathbf{e}_{i_m} \odot_{id} \mathbf{e}_{i_n}) \otimes \mathbf{e}_{i_{m+1}} \otimes \cdots \otimes \mathbf{e}_{i_{n-1}} \otimes \mathbf{e}_{i_{n+1}} \otimes \cdots \otimes \mathbf{e}_{i_k})$. In particular, if $\boldsymbol{\mathcal{A}} = \sum_{i,j \in [d]} \boldsymbol{\mathcal{A}}_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j$ is a 2-order tensor over $\mathbb{K}^d$ (i.e. a square matrix), $\mathbf{v} = \sum \boldsymbol{\mathcal{A}}_{i,j} \mathbf{e}_i \odot_{id} \mathbf{e}_j$ is the diagonal vector of $\boldsymbol{\mathcal{A}}$ and $\mathcal{C}_{1,2}(\boldsymbol{\mathcal{A}}) = \alpha \cdot \mathbf{v}$ is its trace. Furthermore, if $\boldsymbol{\mathcal{A}} = \sum_{i,j \in [d]} \boldsymbol{\mathcal{A}}_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j$ and $\boldsymbol{\mathcal{B}} = \sum_{i,j \in [d]} \boldsymbol{\mathcal{B}}_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j$ are 2-order tensors over $\mathbb{K}^d$, then $\mathcal{C}_{2,3}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{B}}) = \alpha \cdot_2 \left(\sum_{i,j,k,l} \boldsymbol{\mathcal{A}}_{i,j} \boldsymbol{\mathcal{B}}_{k,l} \mathbf{e}_i \otimes (\mathbf{e}_j \odot_{id} \mathbf{e}_k) \otimes \mathbf{e}_l \right)$ is the tensor form of the matrix product $\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{B}}$.

2.3 Hypergraphs

Definition 3. *A hypergraph $G = (V, E, l)$ over a positive ranked alphabet $(\Sigma, \sharp)$ is given by a non empty finite set V, a mapping $l : V \to \Sigma$ and a partition $E = (h_k)_{1 \leq k \leq n_E}$ of $P_G = \{(v, j) : v \in V, 1 \leq j \leq \sharp v\}$ where $\sharp v = \sharp l(v)$.*

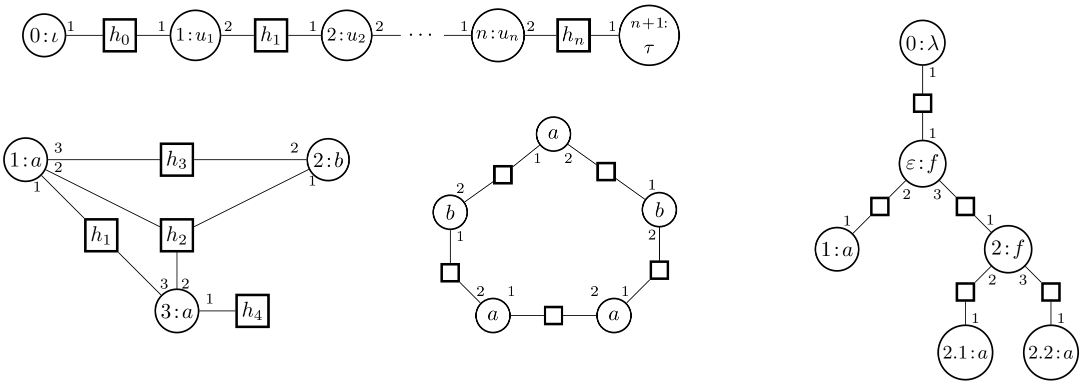

Figure 1: (top) Graph associated with a string $u = u_1 \cdots u_n$. (left) The hypergraph from Example 1. (center) Example of circular string on the alphabet $\{a, b\}$. (right) Hypergraph G_t associated with the tree $t = f(a, f(a, a))$

V is the set of vertices, P_G is the set of ports and E is the set of hyperedges of G. The arity of a symbol x can be seen as the number of *ports* of any vertex labelled by x. We will sometimes use the notation $v^{(i)}$ for the port $(v, i) \in P_G$. A hypergraph is *connected* if for any partition $V = V_1 \cup V_2$, there exists a hyperedge $h \in E$ and ports $v_1^{(i)}, v_2^{(j)} \in h$ s.t. $v_1 \in V_1$ and $v_2 \in V_2$.

Example 1. *Over the ranked alphabet* $\{(a, 3), (b, 2)\}$, *let* $V = \{v_1, v_2, v_3\}$, $l(v_1) = l(v_3) = a$, $l(v_2) = b$, $E = \{h_1, h_2, h_3, h_4\}$ *where* $h_1 = \{v_1^{(1)}, v_3^{(3)}\}$, $h_2 = \{v_1^{(2)}, v_2^{(1)}, v_3^{(2)}\}$, $h_3 = \{v_1^{(3)}, v_2^{(2)}\}$ *and* $h_4 = \{v_3^{(1)}\}$ *(see Figure 1 (left)).*

Example 2. *A string* $u = u_1 \dots u_n$ *over an alphabet* Σ *can be seen as a hypergraph over a ranked alphabet* $(\Sigma \cup \{\iota, \tau\}, \sharp)$ *where* $\sharp x = 2$ *for any* $x \in \Sigma$ *and* $\sharp\iota = \sharp\tau = 1$. *Let* $V = \{0, \cdots, n+1\}$, $l(0) = \iota$, $l(n+1) = \tau$ *and* $l(i) = u_i$ *for* $1 \leq i \leq n$. *Let* $E = \{h_0, h_1, \dots, h_n\}$ *where* $h_0 = \{(0, 1), (1, 1)\}$ *and* $h_i = \{(i, 2), (i+1, 1)\}$ *for* $1 \leq i \leq n$ *(see Figure 1 (top)). The set of strings* Σ^* *gives rise to a family of hypergraphs.*

Example 3. *Similarly, we can associate any tree* t *over a ranked alphabet* $(\Sigma, \sharp)$ *with a graph* G_t *on the ranked alphabet* $(\Sigma \cup \{\lambda\}, \sharp')$ *where* $\sharp'(f) = \sharp f + 1$ *for any* $f \in \Sigma$, *and where the special symbol* λ *of arity 1 is connected to the free port of the vertex corresponding to the root of* t. *The explicit construction of* G_t *can be found in [1], and the graph associated with the tree* $t = f(a, f(a, a))$ *is shown as an example in Figure 1 (right).*

Example 4. *Given a finite alphabet* Σ, *let* $\mathcal{F} = (\Sigma, \sharp)$ *be the ranked alphabet where* $\sharp x = 2$ *for each* $x \in \Sigma$. *We say that a hypergraph* $G = (V, E)$ *on* $\mathcal{F}$ *is a circular string if and only if* G *is connected and every hyperedge* $h \in E$ *is of the form* $h = \{(v, 2), (w, 1)\}$ *for* $v, w \in V$ *(see Figure 1 (middle).*

3 Hypergraph Weighted Models

3.1 Definition

In this section, we give the formal definition of Hypergraph Weighted Models.

Definition 4. *A rank* d *Hypergraph Weighted Model (HWM) on a ranked alphabet* $(\Sigma, \sharp)$ *is a tuple* $M = \langle V_M, \{\mathbf{\mathfrak{T}}^x\}_{x \in \Sigma}, \odot, \alpha \rangle$ *where* $V_M = \mathbb{K}^d$, $\odot$ *is a product on* V_M, $\alpha \in V_M$, *and* $\{\mathbf{\mathfrak{T}}^x\}_{x \in \Sigma}$ *is a family of tensors where each* $\mathbf{\mathfrak{T}}^x \in \bigotimes^{\sharp x} V_M$.

Let $G = (V, E, l)$ be a hypergraph and let $\Gamma = [d]^{P_G}$, the set of mappings from P_G to $[d]$. The series r_M *computed by the HWM M is defined by*

$$r_M(G) = \sum_{\gamma\in\Gamma} \mathfrak{T}_\gamma \prod_{h\in E} \alpha^\top \bigodot_{i\in\gamma(h)} e_i$$

where $\mathfrak{T}_\gamma = \prod_{v\in V} \mathfrak{T}^v_{\gamma(v^{(1)})\dots\gamma(v^{(\sharp v)})}$ (using the notation $\mathfrak{T}^v = \mathfrak{T}^{l(v)}$).

Let $V = \{v_1, \cdots, v_n\}$. The tensor $\mathfrak{T}^{v_1}\otimes\mathfrak{T}^{v_2}\otimes\cdots\otimes\mathfrak{T}^{v_n}$ is of order $|P_G|$, and any element $\gamma \in \Gamma$ can be seen as a multi-index of $[d]^{|P_G|}$. Thus, $\mathfrak{T}_\gamma$ is the $\gamma(v_1^{(1)}), \cdots, \gamma(v_1^{(\sharp v_1)}), \cdots, \gamma(v_n^{(1)}), \cdots, \gamma(v_n^{(\sharp v_n)})$-coordinate of the tensor $\bigotimes_{i=1}^n \mathfrak{T}^{v_i}$.

Example 5. *Consider the hypergraph G from Example 1. We have*
$r_M(G) = \sum_{i_1,\cdots,i_8} \mathfrak{T}^a_{i_1i_2i_3}\mathfrak{T}^b_{i_4i_5}\mathfrak{T}^a_{i_6i_7i_8}\alpha^\top(\mathbf{e}_{i_1}\odot\mathbf{e}_{i_8})\alpha^\top(\mathbf{e}_{i_2}\odot\mathbf{e}_{i_4}\odot\mathbf{e}_{i_7})\alpha^\top(\mathbf{e}_{i_3}\odot\mathbf{e}_{i_5})\alpha^\top\mathbf{e}_{i_6}$.

Remark 2. *If $\odot = \odot_{id}$ and if $\alpha = \mathbf{1}$, then $r_M(G) = \sum_{\gamma\in\Gamma_{Id}} \mathfrak{T}_\gamma$ where $\Gamma_{Id} = \{\gamma \in \Gamma : \forall h \in E, p, q \in h \Rightarrow \gamma(p) = \gamma(q)\}$. For the hypergraph G from Example 1, we would have $r_M(G) = \sum_{i_1,i_2,i_3,i_6} \mathfrak{T}^a_{i_1i_2i_3}\mathfrak{T}^b_{i_2i_3}\mathfrak{T}^a_{i_6i_2i_1}$.*

Remark 3. *Let Σ be a finite alphabet, let $\mathbf{M}_\sigma \in \mathbb{K}^{d\times d}$ for $\sigma\in\Sigma$ and let $A = \langle\mathbb{K}^d, \{\mathbf{M}_\sigma\}_{\sigma\in\Sigma}, \odot_{id}, \mathbf{1}\rangle$ be a HWM. For any non empty word $w = w_1\cdots w_n \in \Sigma^*$ and its corresponding circular string G_w, we have $r_A(G_w) = Tr(\mathbf{M}_{w_1}\cdots\mathbf{M}_{w_n})$ (where $Tr(\mathbf{M})$ is the trace of the matrix $\mathbf{M}$).*

Remark 4. *Note that if G is a hypergraph with two connected components G_1 and G_2, we have $r_M(G) = r_M(G_1)\cdot r_M(G_2)$ for any HWM M.*

Definition 5. *Let $\mathcal{H}$ be a family of hypergraphs on a ranked alphabet $(\Sigma, \sharp)$. We say that a hypergraph series $r : \mathcal{H} \to \mathbb{K}$ is recognizable if and only if there exists a HWM M such that $r_M(G) = r(G)$ for all $G \in \mathcal{H}$.*

3.2 Properties

Propositions 1 and 2 show how HWMs extend linear representations on strings and trees.

Proposition 1. *Let $r : \langle V, \boldsymbol{\iota}, \{\mathbf{M}^\sigma\}_{\sigma\in\Sigma}, \boldsymbol{\tau}\rangle$ be a rational series on Σ^*. For any word $w \in \Sigma^*$, let G_w be the associated hypergraph on the ranked alphabet $(\Sigma\cup\{\iota,\tau\}, \sharp)$, whose construction is described in Example 2. Consider the HWM $M : \langle V, \{\mathfrak{T}^x\}_{x\in\Sigma\cup\{\iota,\tau\}}, \odot_{id}, \mathbf{1}\rangle$ where $\mathfrak{T}^\tau = \boldsymbol{\tau}$, $\mathfrak{T}^\iota = \boldsymbol{\iota}$ and $\mathfrak{T}^\sigma = \mathbf{M}^\sigma$ for all $\sigma\in\Sigma$. Then, $r(w) = r_M(G_w)$ for all strings $w\in\Sigma^*$.*

Proposition 2. *Let $r : \langle V, \mu, \lambda\rangle$ be a rational series on trees on the ranked alphabet $\mathcal{F} = (\Sigma, \sharp)$. For any tree t over $\mathcal{F}$, let G_t be the associated hypergraph on the ranked alphabet $(\Sigma\cup\{\lambda\}, \sharp')$ (see Example 3). There exist a HWM M such that $r_M(G_t) = r(t)$ for any tree t over $\mathcal{F}$.*

The following propositions show that the set of HWMs is closed under addition and Hadamard product.

Proposition 3. *Let $A = \langle\mathbb{K}^m, \{\boldsymbol{\mathcal{A}}^x\}_{x\in\Sigma}, \odot_A, \alpha\rangle$, and $B = \langle\mathbb{K}^n, \{\boldsymbol{\mathcal{B}}^x\}_{x\in\Sigma}, \odot_B, \beta\rangle$ be two HWMs. Define the HWM $C = \langle\mathbb{K}^{m+n}, \{\mathfrak{C}^x\}_{x\in\Sigma}, \odot, \tau\rangle$ by $\tau_i = \alpha_i$ if $1\le i\le m$ and β_{i-m} otherwise,*

$$\mathfrak{C}^x_{i_1\dots i_{\sharp x}} = \begin{cases} \boldsymbol{\mathcal{A}}^x_{i_1\dots i_{\sharp x}} & \text{if } 1 \le i_1,\dots,i_{\sharp x} \le m \\ \boldsymbol{\mathcal{B}}^x_{j_1\dots j_{\sharp x}} & \text{if } m < i_1,\dots,i_{\sharp x} \le m+n \text{ where } j_k = i_k - m \text{ for any } k, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{e}_i \odot \mathbf{e}_j = \begin{cases} \mathbf{e}_i \odot_A \mathbf{e}_j & \text{if } 1 \le i,j \le m \\ t_m(\mathbf{e}_{i-m} \odot_B \mathbf{e}_{j-m}) & \text{if } m < i,j \le n \\ 0 & \text{otherwise} \end{cases}$$

where $t_m : \mathbb{K}^n \to \mathbb{K}^{m+n}$ is the linear mapping defined by $t_m(\mathbf{e}_k) = \mathbf{e}_{k+m}$ for any $1\le k\le n$.

Then the HWM C computes the series r_{A+B} defined by $r_{A+B}(G) = r_A(G) + r_B(G)$, for any connected hypergraph G.

Proposition 4. *Let* $A = \langle \mathbb{K}^m, \{\boldsymbol{\mathcal{A}}^x\}_{x\in\Sigma}, \odot_A, \boldsymbol{\alpha}\rangle$ *and* $B = \langle \mathbb{K}^n, \{\boldsymbol{\mathcal{B}}^x\}_{x\in\Sigma}, \odot_B, \boldsymbol{\beta}\rangle$ *be two HWMs. Identifying* $\mathbb{K}^m \otimes \mathbb{K}^n$ *with* $\mathbb{K}^{mn}$ *via the mapping* $\mathbf{e}_i \otimes \mathbf{e}_j \mapsto \mathbf{e}_{n(i-1)+j}$, *we define the HWM* $D = \langle \mathbb{K}^m \otimes \mathbb{K}^n, \{\boldsymbol{\mathcal{D}}^x\}_{x\in\Sigma}, \odot, \boldsymbol{\delta}\rangle$ *by* $\boldsymbol{\mathcal{D}}^x = \boldsymbol{\mathcal{A}}^x \otimes \boldsymbol{\mathcal{B}}^x$ *for all* $x \in \Sigma$, $(\mathbf{a}_1 \otimes \mathbf{b}_1) \odot (\mathbf{a}_2 \otimes \mathbf{b}_2) = (\mathbf{a}_1 \odot_A \mathbf{a}_2) \otimes (\mathbf{b}_1 \odot_B \mathbf{b}_2)$ *for all* $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{K}^m$ *and* $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{K}^n$, *and* $\boldsymbol{\delta} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ *(i.e.* $\boldsymbol{\delta}^\top(\mathbf{a} \otimes \mathbf{b}) = (\boldsymbol{\alpha}^\top\mathbf{a})(\boldsymbol{\beta}^\top\mathbf{b})$ *for any* $\mathbf{a} \in \mathbb{K}^m$ *and* $\mathbf{b} \in \mathbb{K}^n$*).*

Let r_A *(resp.* r_B*) be the series computed by* A *(resp. by* B*). Then the HWM* C *computes the series* $r_C(G) = r_A(G)r_B(G)$, *for any hypergraph* G.

4 Recognizability of Finite Support Series

In this section, we show that finite support series (i.e. *polynomes*) are not recognizable in general, but we exhibit a wide class of families of hypergraphs for which they are.

First, we show on a simple example why polynomes are not recognizable for all families of hypergraphs. Consider the family of circular strings over a one letter alphabet $\Sigma = \{a\}$ introduced in Example 4 and Remark 3. The following lemma implies that the series r, defined by $r(G_a) = 1$ and $r(G_{a^k}) = 0$ for all integer $k > 1$, is not recognizable. Indeed, r would be such that $r(G_{a^k}) = Tr(\mathbf{M}_a^k) = 0$ for all $k \geq 2$, but it then follows from Lemma 1 that $r(G_a) = Tr(\mathbf{M}_a) = 0$.

Lemma 1. *Let* $M \in \mathbb{R}^{n\times n}$*. If* $Tr(M^k) = 0$ *for all* $k \geq 2$, *then* $Tr(M) = 0$.

This example illustrates the fact that the computation of a HWM on a hypergraph G is done independently on each hyperedge of G. This implies that if two hypergraphs are not distinguishable by just looking at the ports involved in their hyperedges, the computations of a HWM on these two hypergraphs are strongly dependent. This is clear if we consider a hypergraph G_1 made of two copies of a hypergraph G_2 (i.e. G_1 has two connected components, which are both isomorphic to G_2): we have $r(G_1) = r(G_2)^2$ for any HWM r (see Remark 4).

The following section formally introduces the notion of *tiling* of a hypergraph G and show how this relation between hypergraphs relates to the question of the recognizability of polynomes.

4.1 Tilings

A *tiling* of a hypergraph $\widehat{G}$ is a hypergraph G, built on the same alphabet and made of copies of $\widehat{G}$. More precisely,

Definition 6. *Let* $\widehat{G} = (\widehat{V}, \widehat{E}, \widehat{l})$ *be a hypergraph over a ranked alphabet* $(\Sigma, \sharp)$*. A hypergraph* $G = (V, E, l)$ *on the same alphabet* $(\Sigma, \sharp)$ *is a* tiling *of* $\widehat{G}$ *if and only if there exists a mapping* $f : V \to \widehat{V}$ *such that (i)* $l(v) = \widehat{l}(f(v))$ *for any* $v \in V$ *and (ii) the mapping* $g : P_G \to P_{\widehat{G}}$ *defined by* $g(v, i) = (f(v), i)$ *is such that for all* $h \in E$: $g(h) \in \widehat{E}$ *and the restriction* $g_{|h}$ *of* g *to* h *is bijective.*

The following proposition shows that for a connected hypergraph, this formal definition of tiling is equivalent to the intuition of a hypergraph made of copies of the original one.

Let $G = (V, E, l)$ be a tiling of the connected hypergraph $\widehat{G} = (\widehat{V}, \widehat{E}, \widehat{l})$, let $\sim_V$ be the equivalence relation defined on V by $v \sim_V v'$ iff $f(v) = f(v')$, and let $\sim_E$ be the equivalence relation defined on E by $h \sim_E h'$ iff $g(h) = g(h')$ where f and g are the mappings defined above. Clearly, $v \sim_V v'$ entails that $l(v) = l(v')$ and it can easily be shown that $h \sim_E h'$ iff $\exists v^{(i)} \in h, v'^{(i)} \in h'$ such that $v \sim_V v'$. We can thus define the quotient hypergraph $\overline{G} = (V/\sim_v, E/\sim_E, l)$.

Proposition 5. *If* $G = (V, E, l)$ *is a tiling of a connected hypergraph* $\widehat{G} = (\widehat{V}, \widehat{E}, \widehat{l})$, *then* $\overline{G} = (V/\sim_V, E/\sim_E, l)$ *is isomorphic to* $\widehat{G}$ *and moreover, for any* $\hat{v} \in \widehat{V}$, *the cardinal of* $f^{-1}(\{\hat{v}\})$ *is a constant.*

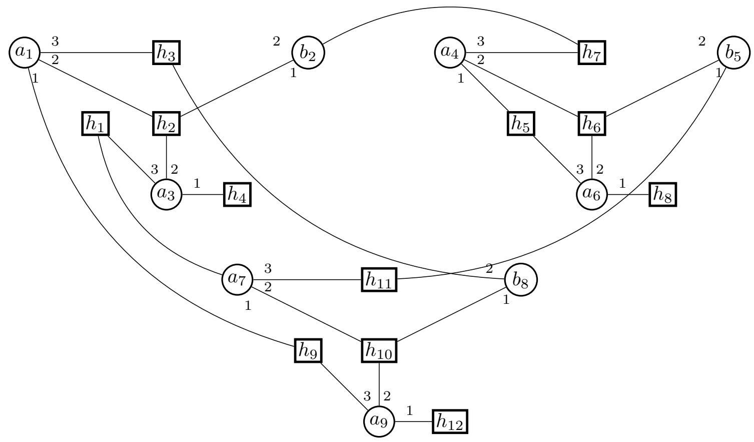

Figure 2: A tiling made of three copies of the hypergraph from Example 1

We end this section with the main result of this paper: there exists a HWM which assigns a nonzero value to a specific hypergraph over some ranked alphabet and all of its tilings, and zero to any other hypergraph on the same alphabet. This result leads to a sufficient condition on families of hypergraphs for the recognizability of finite support series.

Theorem 1. *Given a hypergraph $\widehat{G} = (\widehat{V}, \widehat{E}, \widehat{l})$ over $(\Sigma, \sharp)$, there exists a recognizable series $r_{\widehat{G}}$ such that $r_{\widehat{G}}(G) \neq 0$ if and only if G is a tiling of $\widehat{G}$.*

A family $\mathcal{H}$ of hypergraphs is *tiling-free* if and only if for any $G \in \mathcal{H}$, there are no (non-trivial) tiling of G in $\mathcal{H}$.

Corollary 1. *For any tiling-free family of hypergraphs $\mathcal{H}$, finite support series on $\mathcal{H}$ are recognizable.*

An example of tiling-free family is the family of rooted hypergraphs: hypergraphs on a ranked alphabet $(\Sigma \cup \{\lambda\}, \sharp)$, where the special *root* symbol λ appears exactly once. Some illustrations of the expressiveness of HWM's can be found in [1].

5 Conclusion

The model we propose naturally generalizes rational series on strings and trees. It satisfies closure properties by sum and Hadamard product. We have analysed why finite support series on some families of hypergraphs are not recognizable, and we exhibit a sufficient condition on families of hypergraph for the recognizability of finite support series.

These results suggest that the notion of HWM naturally extends the notion of linear representation for strings and trees, and that the set of recognizable series could be a natural extension of rational series to hypergraphs.

We plan to study how techniques and methods developed in the field of graphical models, such as message passing, variational methods, etc, could be adapted to the setting of HWM. The question of learning HWM from samples also emerges naturally, and could be relevant to the data mining community [5]. Learning algorithms should rely on tensor decomposition techniques, which generalize the spectral methods used for learning rational series on strings and trees. This is a work in progress.

References

[1] Raphaël Bailly, François Denis, and Guillaume Rabusseau. Recognizable series on hypergraphs. *CoRR*, abs/1404.7533, 2014.

[2] Raphaël Bailly, François Denis, and Liva Ralaivola. Grammatical inference as a principal component analysis problem. In *Proceedings of the 26th Annual International Conference on Machine Learning*, ICML '09, pages 33–40, New York, NY, USA, 2009. ACM.

[3] J. Berstel and C. Reutenauer. *Rational series and their languages*. EATCS monographs on theoretical computer science. Springer-Verlag, 1988.

[4] Jean Berstel and Christophe Reutenauer. Recognizable formal power series on trees. *Theoret. Comput. Sci.*, 18(2):115–148, 1982.

[5] A Cichocki. Era of big data processing: A new approach via tensor networks and tensor decompositions. In *International Workshop on Smart Info-Media Systems in Asia*, 2013.

[6] H. Comon, M. Dauchet, R. Gilleron, C. Löding, F. Jacquemard, D. Lugiez, S. Tison, and M. Tommasi. Tree automata techniques and applications, 2007. release October, 12th 2007.

[7] François Denis and Amaury Habrard. Learning rational stochastic tree languages. In *Proceedings of the 18th international conference on Algorithmic Learning Theory*, ALT '07, pages 242–256, Berlin, Heidelberg, 2007. Springer-Verlag.

[8] Wolfgang Hackbusch. *Tensor spaces and numerical tensor calculus*, volume 42 of *Springer series in computational mathematics*. Springer, Heidelberg, 2012.

[9] Roman Orus. A practical introduction to tensor networks: Matrix product states and projected entangled pair states. 2013.