Symbolic Dynamics of Parallel Systems

Wit Foryś*and Magdalena Foryś†

April 2014

Abstract

The aim of this paper is to formulate a framework for studying dynamical properties of parallel computational systems represented by a continuous map acting on a space of infinite traces. We study interrelations between the shifts on infinite words and shifts on infinite real traces as there is no simple analogy between those two types of spaces. In particular we focus on trace shifts generated by minimal shifts and substitution shifts trying to establish conditions which assures us that an image of a shift is a trace shift. We also study the dynamical properties of trace shifts such as mixing, transitivity, shadowing property, expansiveness.

1 Introduction

This paper presents results and problems of symbolic dynamics of parallel or concurrent systems. In [3] we formulated a framework which allows to study parallel dynamics by tools used in a research of sequential symbolic dynamics. To be more precise, in place of infinite words we consider infinite traces introduced in [1, 9] and define a trace shift as a parallel counterpart of a shift - a mathematical model of symbolic dynamics.

In the subsequent sections we present the basic definitions and notations which consists on a framework of considerations, main results in selected fields of research and finally some new results and problems that are the subject of our current study.

2 Definitions and notations

Now we recall only basic concepts of graph theory, symbolic dynamics and theory of traces. All the missing notions may be found in [2, 8].

We consider only simple and finite graphs. By *a co-graph* we understand either a single vertex graph, or the disjoint union of two co-graphs, or the edge complement of a co-graph.

Let Σ be any finite set (alphabet) and denote by Σ^* and Σ^ω the set of all finite and infinite words over Σ, respectively. The set Σ^* with concatenation of words and the empty word, denoted 1 is a free monoid. The set of nonempty words is denoted by Σ^+ and $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$.

Let $I \subset \Sigma \times \Sigma$ be a symmetric and irreflexive relation. In the sequel I is called *an independence (commutation) relation*; its complement is denoted by D and called *a dependence relation.* For every letter $a \in \Sigma$ we denote by $D(a) = \{b \in \Sigma \ : \ (a, b) \notin I\}$, a set of all letters from Σ

*Corresponding author, Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland, forysw@ii.uj.edu.pl

†Faculty of Applied Mathematics, AGH University of Science and Technology, al.Mickiewicza 30, 30-059 Kraków, Poland, maforys@agh.edu.pl

which depend on a. The relation I may be extended to a congruence $\sim_I$ on Σ^*. We have $u \sim_I v$ if and only if it is possible to transform u to v by a finite number of swaps $ab \to ba$ of independent letters. *A trace* is an element of the quotient space $\mathbb{M}(\Sigma, I) = \Sigma^*/\sim_I$. For $w \in \Sigma^*$, $t \in \mathbb{M}(\Sigma, I)$ we denote by $\mid w \mid_a$ and $\mid t \mid_a$ the number of occurrences of the letter $a \in \Sigma$ in w and t respectively. $alph(w)$ and $alph(t)$ denote the set of all letters which occur in w, t. Two traces t_1 and t_2 are *independent*, denoted $t_1 I t_2$, if and only if $alph(t_1) \times alph(t_2) \subset I$. If $x \in \Sigma^\omega$ and $i \leq j$ are nonnegative integers then we denote $x_{[i,j]} = x_i x_{i+1} \ldots x_j$ and $x_{[i,j)} = x_{[i,j-1]}$.

We recall that a word $w \in \Sigma^*$ is in the *Foata normal form*, if it is the empty word or if there exist an integer $n > 0$ and nonempty words $v_1, ..., v_n \in \Sigma^+$ (called *Foata steps*) such that:

1. $w = v_1 .... . v_n$,
2. for any $i = 1, ..., n$ the word v_i is a concatenation of pairwise independent letters and is minimal with respect to the lexicographic ordering,
3. for any $i = 1, ..., n-1$ and for an arbitrary letter $a \in alph(v_{i+1})$ there exists a letter $b \in alph(v_i)$ such that $(a, b) \in D$.

It is well known that for any $x \in \Sigma^*$ there exists the unique $w \in [x]_{\sim_I}$ in the Foata normal form.

In the theory of dynamical systems continuous maps acting on metric spaces are considered. Hence we endow Σ^ω with the following metric d. If $x = y$ then $d(x, y) = 0$ and otherwise $d(x, y) = 2^{-j}$ where j is the number of letters in the longest common prefix of x and y. Now, define a shift map $\sigma : \Sigma^\omega \to \Sigma^\omega$ by

$$(\sigma(x))_i = x_{i+1}$$

where $(\cdot)_i$ denotes the i-th letter of a sequence. It is easy to observe that σ is continuous. Σ^ω together with the map σ is referred to as the *full shift* over Σ. Any closed and σ-invariant (i.e. $\sigma(X) \subset X$) set $X \subset \Sigma$ is called a *shift* or a *subshift*.

For any word $w = (w_i)_{i \in \mathbb{N}} \in \Sigma^\omega$ the *dependence graph* $\varphi_{\mathbb{G}}(w) = [V, E, \lambda]$ is defined as follows. We put $V = \mathbb{N}$ and $\lambda(i) = w_i$ for any $i \in \mathbb{N}$. The function λ successively labels nodes of $\varphi_{\mathbb{G}}(w)$ by letters of w. There exists an arrow $(i, j) \in E$, if and only if $i < j$ and $(w_i, w_j) \in D$. Let us denote the set of all possible dependence graphs (up to an isomorphism of graphs) by $\mathbb{R}^\omega(\Sigma, I)$ and let $\varphi_{\mathbb{G}} : \Sigma^\omega \to \mathbb{R}^\omega(\Sigma, I)$ be a natural projection. We call elements of $\mathbb{R}^\omega(\Sigma, I)$ *infinite (real) traces*. Each dependence graph is acyclic and it induces a well-founded ordering on $\mathbb{N}$. Then for any $v \in V$ the function $h : V \to \mathbb{N}$ given by $h(v) = \max \mathcal{P}(v)$ where
$\mathcal{P}(v) = \{n \in \mathbb{N} : \exists v_1, .., v_n \in V, v_n = v, (v_i, v_{i+1}) \in E \text{ for } i = 1, ..., n-1\}$ is well defined.

By $F_n(t)$ we denote a word $w \in \Sigma^*$ consisting of all the letters from the n-th level of infinite trace $t \in \mathbb{R}^\omega(\Sigma, I)$, that is from the set $\{\lambda(v) : v \in V, h(v) = n\}$. It follows from the definition of a dependence relation that for any infinite trace $t \in \mathbb{R}^\omega(\Sigma, I)$ the word $w = F_1(t) \ldots F_n(t)$ is in the Foata normal form with Foata steps given by $F_i(t)$ and $t = \varphi_{\mathbb{G}}(F_1(t)F_2(t) \ldots)$. Then in the same way as it was done for Σ^ω we may endow $\mathbb{R}^\omega(\Sigma, I)$ with a metric $d_{\mathbb{R}}(s, t)$ putting $d_{\mathbb{R}}(s, t) = 0$ if $s = t$ and $d_{\mathbb{R}}(s, t) = 2^{-j+1}$ if $s \neq t$ where j is the maximal integer such that $F_i(t) = F_i(s)$ for $1 \leq i \leq j$. By a *full t-shift* we mean the metric space $(\mathbb{R}^\omega(\Sigma, I), d_{\mathbb{R}})$ together with a continuous map $\Phi : \mathbb{R}^\omega(\Sigma, I) \to \mathbb{R}^\omega(\Sigma, I)$ defined by the formula $\Phi(t) = \varphi_{\mathbb{G}}(F_2(t)F_3(t) \ldots)$ for any $t \in \mathbb{R}^\omega(\Sigma, I)$. Analogically as a shift is defined, by a *t-shift* we mean any closed and Φ-invariant subset of $\mathbb{R}^\omega(\Sigma, I)$. It was proved in [3] that from a dynamical system point of view $(\mathbb{R}^\omega(\Sigma, I), \Phi)$ is equivalent to a shift of finite type (which means that dynamics of $(\mathbb{R}^\omega(\Sigma, I), \Phi)$ and (Σ^ω, σ) is to some extent similar). However, it frequently happens that the $\varphi_{\mathbb{G}}$ image of a shift is not a t-shift and there are also t-shifts which cannot be obtained as images of any (sequential) shift.

3 Review of the obtained results

This section contains a review of results of our research. Notice that all problems which we consider are of the dual nature – combinatorial (combinatorics on words) and topological one. After formulating a framework which allows to study parallel dynamics by tools of sequential symbolic dynamics we study the following problem.
Find relations between the dynamics of sequential computations and their parallel (trace) counterparts.
Basic observations lead to the following facts. The map $\varphi_{\mathbb{G}}$ which transfers infinite sequences to labelled, directed and acyclic graphs is not continuous, converting a sequential shift (X, σ) by $\varphi_{\mathbb{G}}$ not necessary results in a t-shift and if (Y, Φ) is a t-shift then it may happened that it exists no shift (X, σ) such that $\varphi_{\mathbb{G}}(X) = Y$.

Example 1 – *$\varphi_{\mathbb{G}}$ not continuous.*

Let $\mathcal{A} = \{a, b\}$ and let $(a, b) \in I$. Let us define a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{A}^{\mathbb{N}}$, where $x_n = aba^n ba^\infty$. It is easily seen that
$\lim_{n\to\infty} \varphi_{\mathbb{G}}(x_n) = \lim_{n\to\infty} \varphi_{\mathbb{G}}(ababa^\infty) = \varphi_{\mathbb{G}}(ababa^\infty) \neq \varphi_{\mathbb{G}}(\lim_{n\to\infty} x_n) = \varphi_{\mathbb{G}}(aba^\infty)$

Example 2 — *Converting a shift into t−shift.*

Let $\mathcal{A} = \{a, b\}$ and let $(a, b) \in I$. We define a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{A}^{\mathbb{N}}$ by $x_n = (ba^n)^n a^\infty$ and $X = \mathrm{cl}\left(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \{\sigma^i(x_n)\}\right)$
where cl denotes the closure in $\mathcal{A}^{\mathbb{N}}$. Note that if $y \in X$ and $|y|_b > k$ then between two occurrences of the symbol b the word a^k has to appear. Notice also that if $y \in X$ then $|y|_b < +\infty$. Let $\widetilde{X} = \varphi_{\mathbb{G}}(X)$. Observe that $\varphi_{\mathbb{G}}((ab)^k a^\infty) \in \widetilde{X}$ for every integer k and $\varphi_{\mathbb{G}}((ab)^\infty) \notin \widetilde{X}$ which implies that $\widetilde{X}$ is not closed.

Example 3 — *Looking for a shift to obtain a given t−shift.*

Let $\mathcal{A} = \{a, b, c\}$, $(a, b) \in I$ and $(a, c), (b, c) \notin I$. Let us consider a set $\widetilde{X}$ defined as $\widetilde{X} = \{\varphi_{\mathbb{G}}(abca^\infty), \varphi_{\mathbb{G}}(ca^\infty), \varphi_{\mathbb{G}}(a^\infty)\}$. It is easily seen that $\Phi(\widetilde{X}) \subset \widetilde{X}$ and it is also a closed set. Then $\widetilde{X}$ is t-shift. Now, let us assume that $\widetilde{X} = \varphi_{\mathbb{G}}(X)$ for some shift X. But it follows from the definition of relation I that

$$\varphi_{\mathbb{G}}^{-1}(\varphi_{\mathbb{G}}(abca^\infty)) = \{abca^\infty, baca^\infty\}.$$

The set X is σ-invariant, thus $bca^\infty \in X$ or $aca^\infty \in X$ - a contradiction.

Looking for conditions which assure that for any shift X $\varphi_{\mathbb{G}}(X)$ is closed we find that the following property is essential. Let $(\mathcal{A}, D)$ be a dependence alphabet and X be a shift. If for any positive integer i there exists j such that the following implication holds for any $x, y \in X$

$$x_{[0,j]} = y_{[0,j]} \Rightarrow F_i(\varphi_G(x)) = F_i(\varphi_G(y))$$

then X is said to have a bounded Foata step horizon.

Theorem 4 *Let X be a shift of bounded Foata step horizon. Then $\varphi_{\mathbb{G}}(X)$ is closed.*

Looking further for conditions which assure that $\varphi_{\mathbb{G}}$ image of a shift X is closed and invariant we find that bounded Foata step horizon does not guarantee that invariancy.

Example 5 *Let $\Sigma = \{a, b, c\}$ and $I = \{(a, b), (b, a)\}$. Let $X \subset \Sigma^\omega$ be a minimal shift containing the sequence $p = (aabbc)^\infty$, that is $X = \{p, \ldots, \sigma^4(p)\}$. If we convert X to a subset $\varphi_{\mathbb{G}}(X)$ of infinite traces $\mathbb{R}^\omega(\Sigma, I)$, then it appears that the sequence $(ab)(ab)(c)\ldots$ is in $\varphi_{\mathbb{G}}(X)$, however $(ab)(c)\ldots$ does not belong to this set. Hence $\varphi_{\mathbb{G}}(X)$ is not Φ-invariant.*

The obtained results are listed below.
1. a condition for a $\varphi_{\mathbb{G}}$ image of a shift to be a full $t-$shift (chaos)
2. conditions for a $\varphi_{\mathbb{G}}$ image of a shift to be closed
3. conditions for a $\varphi_{\mathbb{G}}$ image of a shift to be invariant
4. topological (dynamical) properties of $t-$shifts - mixing, transitive, positive topological entropy
5. full t-shift is expansive and has shadowing property.

There were examined also such called dynamical languages of trace shifts. languages composed of finite traces which occur in infinite traces that consist a t-shift.

Theorem 6 *If a trace language is dynamical then the set $X_T \subset \mathbb{R}^\omega(\Sigma, I)$ is a nonempty t-shift and conversely, if $X \subset \mathbb{R}^\omega(\Sigma, I)$ is a t-shift then the trace language T_X associated with it is dynamical.*

Theorem 7 *Let $L \subset \mathcal{A}^*$ be a dynamical language and suppose (X_L, σ) is a shift of finite type. Let us denote $Y = \varphi_{\mathbb{G}}(X_L)$.*
If Y is a t-shift over $(\mathcal{A}, I)$ and for any word $w \in \mathcal{A}^*$ the following condition holds:

$$\varphi_{\mathbb{G}}^{-1}(\varphi_{\mathbb{G}}(w)) \cap L \neq \emptyset \qquad (1)$$

then Y is a full t-shift.

Theorem 8 *A full t-shift over $(\mathcal{A}, D)$ is transitive if and only if the dependence alphabet $(\mathcal{A}, D)$ is represented by a connected graph.*

The second general research problem that was considered was focused on *the class of minimal shifts and minimal t-shifts.* Remind that (X, σ) is a minimal shift if it is closed, nonempty, invariant and contains no proper subset with these three properties. The obtained results are listed below.
1. If X is a minimal shift, then $\varphi_{\mathbb{G}}(X)$ is closed. Even in the simplest case of a minimal system given by a single periodic orbit it may happened that a corresponding set of traces is not invariant.
2. X be a minimal shift, $alph(X) = \Sigma$. Let Σ_1, Σ_2 be a partition of Σ and assume that $\Sigma_1\ D\ \Sigma_2$. Then there exists an integer M such that $Z = \bigcup_{i=0}^{M} \Phi^i(\varphi_{\mathbb{G}}(X)) \quad , \quad Z = \Phi^M(Z)$ are t-shifts. Furthermore $\Phi^M|_Z$ is minimal and $\varphi_{\mathbb{G}}(X) \cap Z \neq \emptyset$
3. We examined in a detailed way $\varphi_{\mathbb{G}}$ images of minimal shifts over small alphabets (up to 3 vertices) and also assuming that an independence relation I is given by a co-graph over 4 vertices.

The next research problem that we considered is devoted to *substitution shifts.* Such systems can be considered as some fundamental bricks helpful for a construction and for the analysis of more complex systems. Remaind that a substitution it is a morphism $h : \Sigma^* \longrightarrow \Sigma^*$. We assume that it satisfies the following conditions (URC):

1. For any letter $a \in \Sigma$ the limit $\lim_{n\to\infty} \mid h^n(a) \mid = +\infty$,
2. there exists a letter $a_0 \in \Sigma$ such that $h(a_0)$ begins with a_0,
3. for any letter $a \in \Sigma$ there exists an integer $k \geq 0$ such that in $h^k(a)$ occurs the letter a_0.

A substitution fulfilling the above conditions defines an infinite word $u = h^\omega(a_0)$ which generates a minimal shift $X = \mathrm{cl}(\mathrm{Orb}^+(u))$.

The obtained results are listed below.

Theorem 9 *If $X = \mathrm{cl}(Orb^+(u))$ is a substitution shift, where $u = h^\omega(a_0)$ for substitution $h : \Sigma^* \longrightarrow \Sigma^*$ fulfilling the conditions (URC), then $\varphi_{\mathbb{G}}(X)$ is closed.*

Even in the class of substitution shifts the invariancy is not behaved. Let us consider the following example.

Example 10 *Let $\Sigma = \{a, b, c, d\}$ and $I = \{(a, b), (b, a)\}$. Let $u = (aabcd)^\infty$ be defined by $h : \Sigma^* \longrightarrow \Sigma^*$ such that $h(a) = aabcd$, $h(b) = h(c) = h(d) = 1$. Hence $X = \mathrm{cl}(Orb^+(u)) = \{u, \sigma(u), \sigma^2(u), \sigma^3(u), \sigma^4(u)\}$. Notice that $\varphi_{\mathbb{G}}(X)$ is not invariant because $\Phi(\varphi_{\mathbb{G}}(u)) = \Phi(\varphi_{\mathbb{G}}(baacd(aabcd)^\infty)) = \varphi_{\mathbb{G}}(acd(aabcd)^\infty) \notin \varphi_{\mathbb{G}}(X)$.*

Theorem 11 *Let $h : \Sigma^* \longrightarrow \Sigma^*$ be a substitution fulfilling the conditions (URC) and $X = \mathrm{cl}(Orb^+(u))$ denotes the substitution shift associated with $u = h^\omega(a_0)$. Let us assume that for any $a_i, a_j \in \Sigma$, $\varphi_{\mathbb{G}}(h(a_i)) = v_1^i v_2^i..v_{r_i}^i$ $\varphi_{\mathbb{G}}(h(a_j)) = v_1^j v_2^j..v_{r_j}^j$ where each v_k^l denote a Foata step in the Foata normal form of $\varphi_{\mathbb{G}}(h(a_i))$, $r_i, r_j > 2$ and one of the following conditions hold:*

(a) for any $y \in alph(v_1^j)$ there exists $x \in alph(v_{r_i}^i)$ such that xDy

(b) $v_{r_i}^i \; I \; v_1^j$ and the condition (a) holds for $v_1^j, v_{r_{i-1}}^i$.

Then $\widehat{w} = (v_1^0)^{-1}\varphi_G(u)$ is a minimal point (almost periodic) where $\varphi_G(h(a_0)) = v_1^0...v_{r_0}^0$.

Theorem 12 *Let Σ, Θ be alphabets, $\Theta \subsetneq \Sigma$. Let $h : \Sigma^* \longrightarrow \Sigma^*$ be a substitution and $X = \mathrm{cl}(Orb^+(u))$ substitution shift associated with $u = h^\omega(a_0)$. Define two relations $F = (\{(a, b) \;:\; a \in \Sigma, b \in \Theta\} \cup \{(b, a) \;:\; a \in \Sigma, b \in \Theta\})\backslash\Delta_\Sigma$, $J = I\backslash F$ where $\Delta_\Sigma = \{(a, a) \;:\; a \in \Sigma\}$ and assume that $F \subset I$. Let $\pi : \Sigma^\omega \to (\Sigma \setminus \Theta)^\infty$ be a projection $\pi(a) = \begin{cases} 1 & if \;\; a \in \Theta \\ a & if \;\; a \notin \Theta \end{cases}$*

Then $\pi(X) \subset (\Sigma \setminus \Theta)^\omega$ and there exists a homeomorphism $\eta : \varphi_{\mathbb{G}}(\pi(X)) \to \varphi_{\mathbb{G}}(X)$ commuting with Φ (i.e. $\eta \circ \Phi = \Phi \circ \eta$ with Φ restricted to $\varphi_{\mathbb{G}}(X)$ and $\varphi_{\mathbb{G}}(\pi(X))$, respectively). Furthermore, if $F = I$ then $\varphi_{\mathbb{G}}(X)$ and $\pi(X)$ are t-shift and shift respectively, dynamical systems $(\varphi_{\mathbb{G}}(X), \Phi)$, $(\pi(X), \sigma)$ are conjugate and $(\pi(X), \sigma)$ is minimal.

Theorem 13 *Let $h : \Sigma^* \longrightarrow \Sigma^*$ be a substitution fulfilling the conditions (URC) and $X = \mathrm{cl}(Orb^+(u))$ denotes the substitution shift associated with $u = h^\omega(a_0)$, $alph(X) = \Sigma$. Let Σ_1, Σ_2 be a partition of Σ and assume that $\Sigma_1 \times \Sigma_2 \subset D$. Then there exists an integer M such that the sets*

$$Y = \bigcup_{i=0}^{M} \Phi^i(\varphi_{\mathbb{G}}(X)) \quad , \quad Z = \Phi^M(Y).$$

are t-shifts. Furthermore $\Phi|_Z$ is minimal and $\varphi_{\mathbb{G}}(X) \cap Z \neq \emptyset$.

4 New directions of research - open problems

There are many open questions which occur during the considerations results of which are presented in the above. For example, a view connected with minimal shifts:
— What is the minimal (in the sense of inclusion) t-shift containing $\varphi_{\mathbb{G}}(X)$,
— What are exactly the conditions when the image $\varphi_{\mathbb{G}}(X)$ of a minimal subshift X is contained in a minimal t-shift or becomes minimal after a finite number of iterations,
— What is the class of minimal shifts, which are contained in a minimal t-shift for any choice of the relation I.

There are a lot of other open problems which occur during our research. Nevertheless, we are currently changed, in some sense, the direction of our research starting with infinite trace space and for example working on substitution t-shifts obtained by iterations of trace morphisms. We hope that results of these research will be presented at the conference.

References

[1] P. Cartier and D. Foata, *Problèmes combinatories de commutation et réarrangements*, Lecture Notes in Mathematics. Springer-Verlag, 1969.

[2] V. Diekert and G. Rozenberg, editors, *The book of traces.* World Scientific Publishing Co. Inc., River Edge, NJ, 1995.

[3] W. Foryś and P. Oprocha, *Infinite Traces and Symbolic Dynamics*, Theory Comput Syst., 45 (2009), 133-149.

[4] W. Foryś, J.L.G. Guirao, P. Oprocha, *A dynamical model of parallel computation on bi-infinite time-scale*, J.Comput. Appl. Math., 235 (2011), 1826–1832.

[5] W. Foryś, P. Oprocha, S. Bakalarski, *Symbolic Dynamics, Flower Automata and Infinite Traces*, LNCS 6482 (2011), 135–142.

[6] W. Foryś and P. Oprocha, *Infinite traces and symbolic dynamics - minimal shift case*, Fundamenta Informaticae 111 (2011), 147–161.

[7] W. Foryś, G. Semanišin, M. Foryś *Trace shifts - minimal case for independence relations given by five node co-graphs*, ITAT 2013

[8] D. Lind and B. Marcus, *Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, 1995.

[9] A. Mazurkiewicz, *Concurrent program schemes and their interpretations*, DAIMI Rep. Aarhus University, 78:1–45, 1977.