Tilings by 1×1 and 2×2 squares

Nicolas Rolin   Alexandra Ugolnikova

LIPN, Univ. Paris 13

1 Introduction

In the present work, we look at tilings of a $k \times n$ board ($n, k \in \mathbb{N}$) by 1×1(small) and 2×2 (big) squares with no holes or overlaping. The goal is to understand how the average proportion of small squares in tilings of a $k \times n$ rectangle by small and big squares changes when $k, n \to +\infty$. A simpler problem (and the one we study here) is to consider that k is fixed and $n \to +\infty$.

There has been some work done on the subject. When $k = 2$, tilings of a $2 \times n$ rectangle by 1×1 and 2×2 squares correspond to the Fibonacci sequence. For $k = 3$, one can easily show that the number of ways to cover a $3 \times n$ rectangle with 1×1 and 2×2 squares is equal to $\frac{1}{3}(-1)^n + \frac{1}{3}2^{n+1}$. Explicit formulas for the number of tilings for k up to 5 were obtained by Heubach S. [2, 3] The bigger cases, however, seem to need the use of other methods.

This abstract consists of two main sections. In Section 2 we define a set of Bivariate Generating Functions ($BGFs$) associated with tilings of a $k \times n$ rectangle, present formulas and calculate distribution of small squares in tilings for k up till 10. In Section 3 we introduce an automaton construction that represents $BGFs$ and their relations. We extract some properties on its structure, present a simplification algorithm that allows to find $BGFs$ more easily.

2 Settings, definitions

2.1 Bivariate Generating Functions

In order to study the general case, we introduce $BGFs$. Let us define them for the case $k = 4$ and then generalize the definition. Let

$$Q_{0000}(z, u) = \sum_{n,p} A^4_{n,p} z^n u^p$$

be a BGF where the coefficient of $z^n u^p$ ($A^4_{n,p}$) is the number of tilings of a $4 \times n$ rectangle with exactly p small squares.

Let $Q_{1000}(z, u)$ be a BGF with the coefficient of $z^n u^p$ being the number of tilings of a $4 \times n$ rectangle with a 1×1 square cut off from the upper left corner and $Q_{2200}(z, u)$ a BGF with the coefficient of $z^n u^p$ corresponding to the number of tilings of a $4 \times n$ rectangle with a 2×2 square cut off from the upper left corner (illustrations are shown in Figure 1).

Figure 1: $4 \times n$ board with cut off corners

From this point on, we will write *BGFs* without arguments, always meaning that they are z, u. A relation on Q_{0000}, Q_{1000} and Q_{2200} can be expressed in the following way:

$$Q_{0000} = zuQ_{1000} + z^4 Q_{2200}.$$

Indeed, in order to obtain Q_{0000}, we can either cut off a small square or a big one from the upper left corner. The remaining areas will correspond either to Q_{1000} or Q_{2200}. And because we cut off squares we need to multiply Q_{1000} by zu (z corresponds to the area occupied by a small square, u – to the one small square) and Q_{2200} by z^4 respectively. In the same way we can introduce, e.g., Q_{1100} and Q_{1220}, and a relation on them and Q_{1000}.

At each step we change indexes of $Q_{i_1 i_2 i_3 i_4}$ by going from left to right in the following way: we permit changing either one 0 to 1 or 00 to 22, which means changing the left one or two columns of the board that was obtained at the previous step by cutting off either a 1×1 or a 2×2 square from the upper left corner of the board.

As soon as we get to $Q_{i_1 i_2 i_3 i_4}$ with all indexes being different from zero, we use a *tetris* rule and reduce the indexes of $Q_{i_1 i_2 i_3 i_4}$ by one layer with "no charge". For example, Q_{1122} gets reduced to Q_{0011}, Q_{1111} to Q_{0000} and so on.

Using this technique one obtains a finite set of *BGFs* $Q_{i_1 i_2 i_3 i_4}$ and a system of functional equations on them. For $k \geq 5$, the principle of constructing a set of $Q_{i_1 \ldots i_k}$ and a system of functional equations is the same.

2.2 Combinatorical results

Using traditional combinatorical tools (see, e.g., [1]) we can find formulas for our *BGFs* and extract some properties. We can solve a system of equations and find $Q_{0 \ldots 0}(z, u)$ for small k. It starts getting complex for $k \geq 10$ given that the size of the associated matrix grows exponentially.

For example, for $k = 4$

$$Q_{0000}(z,u) = \frac{1 - z^4}{1 - z^4 - z^4u^4 - 2z^8u^4 - z^8 + z^{12}u^4 + z^{12}}$$

The coefficient of z^n in $Q_{0000}(z,1)$ corresponds to the number tilings of a $4 \times n$ rectangle without restriction on the number of small squares. These coefficients satisfy the recurrence equation: $a_n = 2a_{n-1} + 3a_{n-2} - 2a_{n-3}$ with $a_0 = a_1 = 1, a_2 = 5$ [A054854] [4].

For every $Q_{0\ldots0}(z,u)$ (with k zero indexes) let z_0 be the singularity closest to zero. Then

$$\left.\frac{\partial_u Q_{0\ldots0}(z,u)}{z\partial_z Q_{0\ldots0}(z,u)}\right|_{(1,z_0)}$$

gives us average proportion of space occupied by small squares in tilings of a $k \times n$ rectangle.

Average proportions of space occupied by small squares for $k \leq 10$ are shown in the table below.

k	z_0	%
3	0.7937	55.555
4	0.7721	46.954
5	0.7701	49.507
6	0.7642	47.241
7	0.7621	47.759
8	0.7596	47.029
9	0.7586	47.651
10	0.7656	46.923

The sequence of proportions seems to be converging. The question is, whether it indeed converges and if so, what is its limit?

3 Automaton representation

For each k let us introduce an automaton. Each $Q_{i_1\ldots i_k}$ with $i_j \in \{0,1,2\}$ for $j = 1,\ldots k$ is associated with a state $q = i_1\ldots i_k$ and each functional equation involving *BGFs* is translated into an automaton transition. For example, the relation

$$Q_{0000} = zuQ_{1000} + z^4Q_{2200}.$$

is represented in the following way: an arrow marked by zu goes from the state 1000 to the state 0000, an arrow marked by z^4 goes from the state 2200 to the state 0000. When the tetris rule is applied, we will mark corresponding arrows by *a star*. An illustration of an automaton for $k = 4$ is shown in Figure 2.

Calculation of paths in the automaton that start and end at the state $0\ldots0$ will allow us to find formulas for $Q_{0\ldots0}$. Our objective is to decrease computational complexity by reducing the number of states.

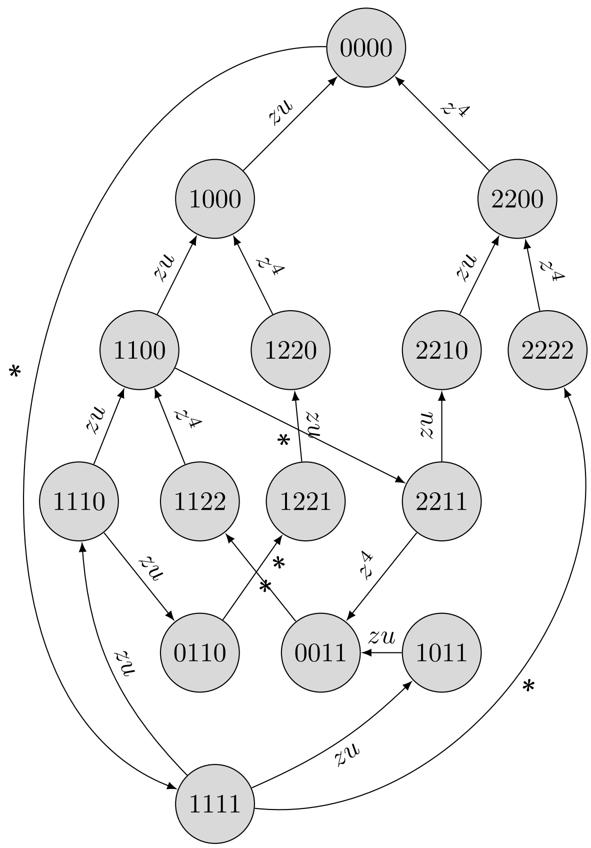

Figure 2: Automaton for $k = 4$

3.1 Essential, non-essential and additional states

Definition 1. *A state q of an automaton is called essential if there are at least two arrows coming in and out of q and at least one of the arrows coming out is marked by a star. It is called non-essential otherwise.*

Note: We consider the state $0\ldots0$ to be essential.

Let E_k be the set of all essential states for each $k \geq 4$. From Figure 2 we get $E_4 = \{0000, 1100\}$ and $\mid E_4 \mid = 2$. Let us describe the structure of E_k and find $\mid E_k \mid$.

Proposition 1. *A state $q = i_1 \ldots i_k$ of an automaton is essential if and only if q has the following propertries:*

1. *q consists only of* 0 *and* 1*;*

2. *All* 1 *come in pairs in q;*

3. $i_1 = i_2 = 1$*;*

4. *The leftmost* 0 *in q comes in pair with another* 0*.*

Commentary: Let us point out one more time that we consider that the state $0 \dots 0$ is always essential.

Proposition 2. *For $k \geq 4$ the number of essential states $\mid E_k \mid$ in the automaton is represented by the following formula:*

$$\mid E_k \mid = \sum_{i=1}^{\lfloor \frac{k-2}{2} \rfloor} \sum_{j=1}^{i} \binom{k-i-j-2}{i-j} + 1.$$

Definition 2. *A state is called additional if it belongs to a cycle that doesn't include any essential state.*

Note that only non-essential states can be additional. The interest of looking at additional states is merely because in order to properly reduce an automaton we need to pay attention to all the cycles in the automaton.

Let us define a subset of the set of additional states that that have the same structure as essential states but with an odd number of 1 on the left from the leftmost 0. We denote this set by A_k^E for $k \geq 5$ and the states by $q_1, \dots, q_N$. It follows from Proposition 2 that $N = \mid E_{k-1} \mid -1$.

Proposition 3. *Each state from A_k^E is additional and generates at least one cycle that doesn't contain any essential state.*

Now the question is, if we mark all the states from E_k and A_k^E in the automaton, will it ensure that there will be no cycles left that don't contain the marked states? Proving that will justify our choice for keeping these particular states.

Proposition 4. *Let $q = i_1 \dots i_k$ be a state of the automaton that doesn't belong to $A_k^E \cup E_k$. If q belongs to a cycle, then this cycle contains states from $A_k^E \cup E_k$.*

We shall further refer to the states from A_k^E as *E-additional* states.

3.2 Simplified automata

We can simplify an automaton by keeping only essential and E-additional states and reducing all other states. The rules of reduction are shown in Figure 3. We denote as f_{ij} a transition between states q_i and q_j which is represented by an arrow going from q_i to q_j.

Using the rules of reduction we obtain an automaton. For each k the result is unique and doesn't depend on the order in which we apply the rules of reduction.

Reduced automata for cases $k = 4, 5$ are shown in Figure 4 and 5.

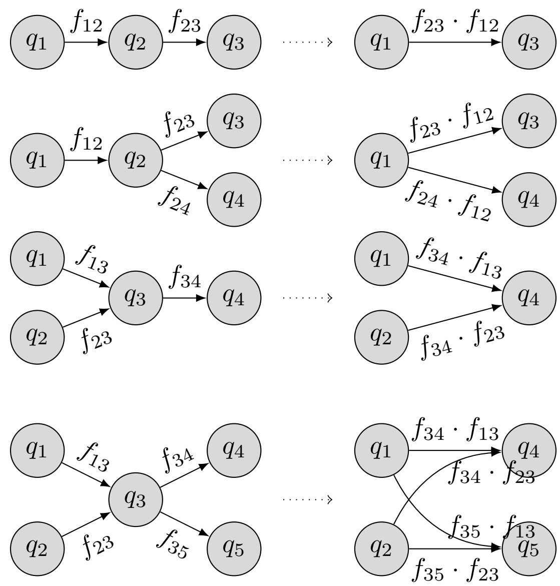

Figure 3: Rules of reduction for an automaton

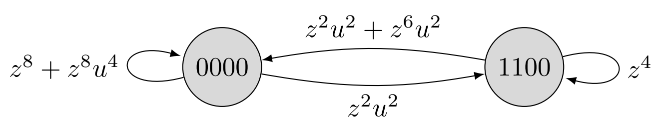

Figure 4: Reduced automaton for $k = 4$

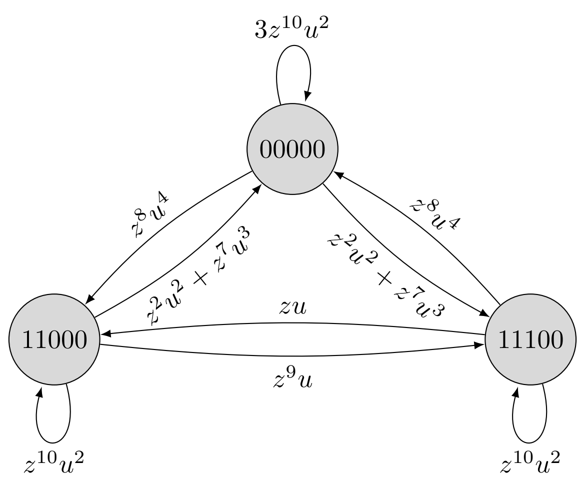

Figure 5: Reduced automaton for $k = 5$

References

[1] Flajolet Ph., Sedgewick S., Analytical Combinatorics, Cambridge University Press, 2009

[2] Heubach S., Tiling an m-by-n Area with Squares of Size up to k-by-k with m $\leq$ 5, Congressus Numerantium 140 (1999), 43-64

[3] Heubach S., Chinn P., Patterns Arising From Tiling Rectangles with 1-by-1 and 2-by-2 Squares, 2000

[4] Sloane N.J.A., and Plouffe S., The Encyclopedia of Integer Sequences, Academic Press Inc., 1995