Beta-Expansions of Rational Numbers in the Quadratic Pisot Bases

Tomáš Hejda,* and Wolfgang Steiner†

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Abstract

We study the purely periodic β-expansions of rational numbers. We give an algorithm for determining the value of the function $\gamma(\beta)$ for quadratic Pisot numbers β. For numbers satisfying $\beta^2 = a\beta + b$ with b dividing a, we show a necessary and sufficient condition for $\gamma(\beta) = 1$, i.e., that all rational numbers $p/q \in [0, 1)$ with $\gcd(q, b) = 1$ have a purely periodic β-expansion.

1 Introduction

Rényi β-expansions [Rén57] provide a very natural generalization of standard positional numeration systems such as the decimal system. Expansions of numbers $x \in [0, 1)$ can be defined in terms of a transformation. Let $\beta > 1$ denote the base. Then the β-transformation is the map

$$T : [0, 1) \to [0, 1),\ x \mapsto \beta x - \lfloor \beta x \rfloor. \tag{1.1}$$

The expansion of x is the infinite string $x_1 x_2 x_3 \cdots$ where $x_j := \lfloor \beta T^{j-1} x \rfloor$. It is a well-known fact that for $\beta \in \mathbb{N}$, the β-expansion of $x \in [0, 1)$ is eventually periodic (i.e., there exists p, n such that $x_{k+p} = x_p$ for all $k \geq n$) if and only if $x \in \mathbb{Q}$. This result was generalized to all Pisot bases by Schmidt [Sch80], who proved that for a Pisot number β the expansion of $x \in [0, 1)$ is eventually periodic if and only if x is an element of the algebraic field $\mathbb{Q}(\beta)$. Moreover, he showed that when β satisfies $\beta^2 = a\beta + 1$, then all $x \in [0, 1) \cap \mathbb{Q}$ have a purely periodic β-expansion.

Akiyama [Aki98] showed that if β is a Pisot unit satisfying a certain finiteness property called (F') then there exists $c > 0$ such that all rational numbers $x \in \mathbb{Q} \cap [0, c)$ have a purely periodic expansion. If β is not a unit, then a rational number $p/q \in [0, 1)$ can have a purely periodic expansion only if q is co-prime to the norm $N(\beta)$. We denote $\mathbb{Z}_b$ the set of rational numbers p/q with $\gcd(q, b) = 1$. Many Pisot non-units satisfy that there exists $c > 0$ such that all $x \in \mathbb{Z}_{N(\beta)} \cap [0, c)$ have purely periodic expansion. This stimulates for the following definition:

Definition 1.1. Let β be a Pisot number, and let $N(\beta)$ denote the norm of β. Then we define $\gamma(\beta) \in [0, 1]$ as the infimum of positive $p/q \in \mathbb{Q}$ with $\gcd(q, N(\beta)) = 1$ and with not purely periodic β-expansion:

$$\gamma(\beta) := \inf\left\{ \tfrac{p}{q} : p, q > 0,\ \gcd(q, N(\beta)) = 1,\ \tfrac{p}{q} \text{ does not have a purely periodic } \beta\text{-expansion} \right\}.$$

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The question is how to determine the value of $\gamma(\beta)$. As well, knowing when $\gamma(\beta) = 0$ or 1 is of big interest.

The transformation T possesses an ergodic invariant measure. Therefore this transformation on the interval $[0, 1)$ forms a dynamical system. It is easy to observe that the expansion of x is purely periodic if and only if x is a periodic point of T, i.e., there exists $p \geq 1$ such that $T^p x = x$. The natural extension $(\mathcal{X}, \mathcal{T})$ of $([0, 1), T)$ can be defined in an algebraic way, cf. (2.1). Taking this form of the natural extension, several authors contributed to proving the following result: A point $x \in [0, 1)$ has purely periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and its diagonal embedding lies in the natural extension domain $\mathcal{X}$. The quadratic unit case was solved by Hama and Imahashi [HI97], the confluent unit case by Ito and Sano [IS01, IS02]. Then Ito and Rao [IR05] resolved the unit case completely using an algebraic argument. For non-unit bases β, one has to consider finite (p-adic) places of the field $\mathbb{Q}(\beta)$. This consideration allowed Berthé and Siegel [BS07] to expand the result to all (non-unit) Pisot numbers.

The first values of $\gamma(\beta)$ for two particular non-units were provided by Akiyama, Barat, Berthé and Siegel [ABBS08]. Recently, Minervino and Steiner [MS14] described the boundary of $\mathcal{X}$ for quadratic non-unit Pisot bases. This allowed them to find the value of $\gamma(\beta)$:

Theorem 1.2 ([MS14]). *Let β be the positive root of $\beta^2 = a\beta + b$ for $a \geq b > 0$ two co-prime integers. Then*

$$\gamma(\beta) = \begin{cases} 1 - \frac{(b-1)b\beta}{\beta^2 - b^2} \in (0, 1) & \text{if } a > b(b-1), \\ 0 & \text{otherwise.} \end{cases}$$

2 Preliminaries

Combinatorics on words. We consider both finite and infinite words over a finite alphabet $\mathcal{A}$. The set of finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$. An infinite word is *(eventually) periodic* if it is of the form $v(u)^\omega = vuuu\cdots$; $v \in \mathcal{A}^*$ is the pre-period and $u \in \mathcal{A}^* \setminus \mathcal{A}^0$ is the period; if the pre-period is empty, we speak about a *purely periodic word.* The set of all infinite words over $\mathcal{A}$ is denoted $\mathcal{A}^\omega$, and it is equipped with the Cantor topology. A prefix of a (finite or infinite) word w is any finite word v such that w can be written as $w = vu$ for some word u. We denote by $\mathrm{Pref}(\Omega)$ for $\Omega \subseteq \mathcal{A}^\omega$ the set of all finite prefixes of words in Ω.

For a finite word $u = u_0u_1 \dots u_{k-1}$ and an arbitrary number α we define a natural polynomial representation of the word as

$$P(\alpha, u) := \sum_{i=0}^{k-1} u_i \alpha^i.$$

This definition is extended to infinite words by taking a limit if the limit exists.

Representation spaces and beta-tiles. We adopt the notation of [MS14], however, we restrict ourselves to β being a quadratic Pisot number. Let $K = \mathbb{Q}(\beta)$. Since β is quadratic, we know that there are exactly two infinite places of K. In one of them, the norm of x is the absolute value $|x|$; in the second one, K', the norm of x is $|x'|$ where $x \to x'$ is the unique non-identical Galois isomorphism of K. Both these places have $\mathbb{R}$ as their completion.

If β is not a unit, then we have to consider finite places of K as well. We put $\mathbb{K}_\mathrm{f} := \prod_{\mathfrak{p}|(\beta)} K_\mathfrak{p}$. The convergence in $\mathbb{K}_\mathrm{f}$ can be expressed in terms of β-adic expansions, cf. §3. Finally, $\mathbb{K} := K \times K' \times \mathbb{K}_\mathrm{f}$ and $\mathbb{K}' := K' \times \mathbb{K}_\mathrm{f}$. We define the diagonal embeddings

$$\delta : \mathbb{Q}(\beta) \to \mathbb{K},\ x \mapsto (x, x', x_\mathrm{f}) \quad \text{and} \quad \delta' : \mathbb{Q}(\beta) \to \mathbb{K}',\ x \mapsto (x', x_\mathrm{f}),$$

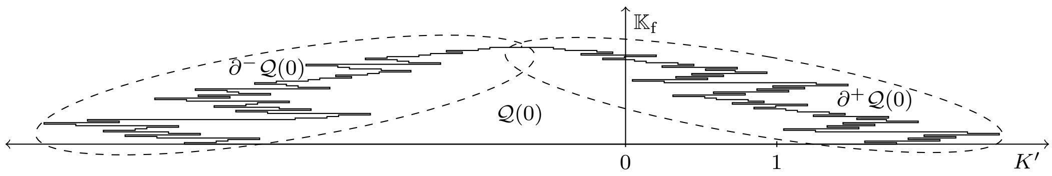

Figure 1: The two boundaries of the tile $\mathcal{Q}(0)$ for $\beta = 1+\sqrt{3}$.

where x_f is the vector of the embeddings of x into the spaces $K_\mathfrak{p}$. As well, we define the projections $\pi_1 : \mathbb{K} \to K$ and $\pi_2 : \mathbb{K} \to K'$. We put $P'(u) = P(\beta', u)$ and $P_\mathrm{f}(u) = P(\beta, u)_\mathrm{f}$ for every word u.

For $x \in [0,1)$, we define the β-tile of x as the Hausdorff limit

$$\mathcal{Q}(x) := \lim_{i\to\infty} \delta'\big(x - \beta^k T^{-k}(x)\big) \subseteq \mathbb{K}'.$$

Note that the standard definition of a β-tile for $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$ is $\mathcal{R}(x) := \delta'(x) - \mathcal{Q}(x)$. We now describe the natural extension $(\mathcal{X}, \mathcal{T})$ of the dynamical system $([0,1), T)$ as a subset of the representation space $\mathbb{K}$. For quadratic Pisot β, root of $\beta^2 = a\beta + b$ with $a \geq b \geq 1$, it comprises of two suspensions of β-tiles:

$$\begin{aligned} &\mathcal{X} := \big([0, \beta - a) \times \mathcal{Q}(0)\big) \cup \big([\beta - a, 1) \times \mathcal{Q}(\beta - a)\big), \\ &\mathcal{T} : \mathcal{X} \to \mathcal{X},\ \boldsymbol{z} \mapsto \beta\boldsymbol{z} - \delta\big(\lfloor \beta\pi_1(\boldsymbol{z}) \rfloor\big). \end{aligned} \tag{2.1}$$

It is remarkable that the natural extension given by this formula is not a closed set, for with the given definition, the following important result holds:

Theorem 2.1 ([HI97, IR05, BS07])**.** *For a Pisot number β, we have that x has a purely periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and $\delta(x) \in \mathcal{X}$.*

3 Hensel expansions of quadratic numbers

Throughout the rest of the paper, we will fix arbitrary quadratic Pisot number β, root of $\beta^2 = a\beta + b$ with $a \geq b \geq 1$. Let $\mathbb{Z}_b := \{p/q : p \in \mathbb{Z},\ \gcd(q, b) = 1\}$ be the set of rational numbers whose denominator is co-prime to b.

The map P_f is a homeomorphism (a bijection that is continuous both ways) from $\mathcal{A}^\omega$ to $\overline{\mathbb{Z}_b[\beta]_\mathrm{f}}$, where the alphabet is $\mathcal{A} := \{0, 1, \ldots, |N(\beta)| - 1\}$. Its inverse is the *Hensel expansion map* $\boldsymbol{h} : \overline{\mathbb{Z}_b[\beta]_\mathrm{f}} \to \mathcal{A}^\omega$, whose fundamental property is that for $x \in \mathbb{Z}_b[\beta]$, the Hensel expansion $\boldsymbol{h}(x) = x_0x_1x_2\cdots$ satisfies that

$$x - \sum_{i=0}^{n} x_i\beta^i \in \beta^n\mathbb{Z}_b[\beta] \quad \text{for all} \quad n \geq 0. \tag{3.1}$$

In §9.3 of the article [MS14], the boundary of β-tiles $\mathcal{Q}(0)$ and $\mathcal{Q}(\beta - a)$ is described. The tiles have two boundaries: $\partial^+\mathcal{Q}(x)$ on the right and $\partial^-\mathcal{Q}(x)$ on the left (see Figure 1). We have

$$\begin{aligned} \partial^+\mathcal{Q}(0) = \partial^+\mathcal{Q}(\beta - a) &= \big\{\big(1 + P'(\boldsymbol{u}), 1_\mathrm{f} + P_\mathrm{f}(\boldsymbol{u})\big) : \boldsymbol{u} \in \mathcal{A}^\omega\big\}, \\ \partial^-\mathcal{Q}(0) &= \big\{\big(\beta' - a + P'(\boldsymbol{u}), \beta_\mathrm{f} - a_\mathrm{f} + P_\mathrm{f}(\boldsymbol{u})\big) : \boldsymbol{u} \in \mathcal{A}^\omega\big\}, \\ \partial^-\mathcal{Q}(\beta - a) &= \big\{\big(\beta' - a + 1 + P'(\boldsymbol{u}), \beta_\mathrm{f} - a_\mathrm{f} + 1_\mathrm{f} + P_\mathrm{f}(\boldsymbol{u})\big) : \boldsymbol{u} \in \mathcal{A}^\omega\big\} \end{aligned} \tag{3.2}$$

(all these sets lie in $\mathbb{K}'$). We can express the value of $\gamma(\beta)$ easily in terms of the boundaries:

Theorem 3.1. *Let β be a quadratic Pisot number. Denote $Y' := K' \times \overline{(\mathbb{Z})_{\mathrm{f}}} \subseteq \mathbb{K}'$ and put*

$$\bar{\gamma} := \inf \pi_2\big(\partial^+ \mathcal{Q}(0) \cap Y'\big). \tag{3.3}$$

If $\sup \pi_2\big(\partial^- \mathcal{Q}(0) \cap Y'\big) > 0$, then $\gamma(\beta) = 0$. Otherwise if $\sup \pi_2\big(\partial^- \mathcal{Q}(\beta - a) \cap Y'\big) > 0$, then $\gamma(\beta) = \min\{\beta - a, \max\{\bar{\gamma}, 0\}\}$. Otherwise $\gamma(\beta) = \max\{\bar{\gamma}, 0\}$.

Remark 3.2. We can change $\mathbb{Z}$ in the statement of the theorem to $\mathbb{Z}_b$ or to $\mathbb{Z}_b \cap [c, d]$ for arbitrary $c < d$ since we have that $\overline{(\mathbb{Z})_{\mathrm{f}}} = \overline{(\mathbb{Z}_b)_{\mathrm{f}}} = \overline{(\mathbb{Z}_b \cap [c, d])_{\mathrm{f}}}$.

For many cases, we obtain the following direct formula:

Corollary 3.3. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ for $a \geq b \geq 1$. Suppose $a > \frac{1+\sqrt{5}}{2} b$ or $a = b$ or $\gcd(a, b) = 1$. Then*

$$\gamma(\beta) = \max\big\{0, \inf \pi_2\big(\partial^+ \mathcal{Q}(x) \cap Y'\big)\big\}. \tag{3.4}$$

For the boundary, we observe the following thing, which follows from the fact that the boundary is continuous as a function from $\overline{\mathbb{Z}[\beta]_{\mathrm{f}}} \to K'$:

Lemma 3.4. *For every $n \in \mathbb{N}$, we have that each of the boundaries $\partial^{\pm}(x)$ for $x \in \{0, \beta - a\}$ is contained in a union of rectangles,*

$$\begin{aligned}
\partial^+ \mathcal{Q}(x) &\subset \bigcup_{w \in \mathcal{A}^n} \big(1 + P'(w) + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]\big) \times \big(1_{\mathrm{f}} + P_{\mathrm{f}}(w\mathcal{A}^{\omega})\big), \\
\partial^- \mathcal{Q}(0) &\subset \bigcup_{w \in \mathcal{A}^n} \big(\beta' - a + P'(w) + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]\big) \times \big(\beta_{\mathrm{f}} - a_{\mathrm{f}} + P_{\mathrm{f}}(w\mathcal{A}^{\omega})\big), \\
\partial^- \mathcal{Q}(\beta - a) &\subset \bigcup_{w \in \mathcal{A}^n} \big(\beta' - a + 1 + P'(w) + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]\big) \times \big(\beta_{\mathrm{f}} - a_{\mathrm{f}} + 1_{\mathrm{f}} + P_{\mathrm{f}}(w\mathcal{A}^{\omega})\big).
\end{aligned} \tag{3.5}$$

Proposition 3.5. *Let $\mathcal{L}_y$ for $y \in \mathbb{Z}[\beta]$ be the language of prefixes of Hensel expansions of numbers from the set $\mathbb{Z} - y$, i.e., $\mathcal{L}_y := \mathrm{Pref}\big\{\boldsymbol{h}(k + y) : k \in \mathbb{Z}\big\}$. Then for each $n \in \mathbb{N}$ and $x \in \{0, \beta - a\}$ we can estimate*

$$\begin{aligned}
\inf \pi_2\big(\partial^+ \mathcal{Q}(x) \cap Y'\big) &\in 1 + \min\big\{P'(w) : w \in \mathcal{L}_0 \cap \mathcal{A}^n\big\} + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1], \\
\sup \pi_2\big(\partial^- \mathcal{Q}(0) \cap Y'\big) &\in \beta' - a + \max\big\{P'(w) : w \in \mathcal{L}_\beta \cap \mathcal{A}^n\big\} + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1], \\
\sup \pi_2\big(\partial^- \mathcal{Q}(\beta - a) \cap Y'\big) &\in \beta' - a + 1 + \max\big\{P'(w) : w \in \mathcal{L}_\beta \cap \mathcal{A}^n\big\} + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1].
\end{aligned} \tag{3.6}$$

The right-hand sides of (3.6) are itervals whose lengths shrink exponentially when $n \to \infty$. The only remaining step is to construct the languages $\mathcal{L}_x \cap \mathcal{A}^n$, which is solved by the following statement:

Proposition 3.6. *Let $x, z \in \mathbb{Z}[\beta]$ satisfy that $x - z \in b^n\mathbb{Z}$ for some $n \in \mathbb{N}$. Then the Hensel expansions $\boldsymbol{h}(x)$ and $\boldsymbol{h}(z)$ have a common prefix of the length at least n.*

Therefore all elements of $\mathcal{L}_y$ of the length n are precisely

$$\mathcal{L}_y \cap \mathcal{A}^n = \mathrm{Pref}\big\{\boldsymbol{h}(y + k) : k \in \{0, 1, \ldots, b^n - 1\}\big\} \cap \mathcal{A}^n, \quad \textit{whence} \quad \#(\mathcal{L}_y \cap \mathcal{A}^n) \leq b^n. \tag{3.7}$$

4 The case b divides a

In the particular case when b divides a, the structure of $\mathcal{L}_y$ is even simpler, namely we have that $\#(\mathcal{L}_y \cap \mathcal{A}^n) = b^{\lceil n/2 \rceil}$, therefore $\#(\mathcal{L}_y \cap \mathcal{A}^{2n}) = \#(\mathcal{L}_y \cap \mathcal{A}^{2n-1})$. This is given by the fact that in this case, $b^k\mathbb{Z}[\beta] = \beta^{2k}\mathbb{Z}[\beta]$. The result for this case can be stated as follows:

$a/b =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$b = 1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	$\star$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	$\star$	1	1	1	1	1	1	1	1	1	1	1	1	1
4	0	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1	1
5	0	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1
6	0	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1	1
7	0	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1	1
8	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1
9	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1
10	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1
11	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1
12	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1

Table 1: The values of $\gamma(\beta)$ for the case when b divides a. The '$\star$' means that the value is strictly between 0 and 1.

a	b	$\gamma(\beta)$
2	2	$0.914803044196\cdots$
6	3	$0.992963560101\cdots$
8	4	$0.933542944675\cdots$
12	4	$0.999897789000\cdots$
10	5	$0.834150794175\cdots$
15	5	$0.995306723671\cdots$
20	5	$0.999999907110\cdots$

a	b	$\gamma(\beta)$
12	6	$0.736114178272\cdots$
18	6	$0.993897266395\cdots$
14	7	$0.584906533458\cdots$
21	7	$0.944526094618\cdots$
28	7	$0.997984788082\cdots$
35	7	$0.999986041767\cdots$
42	7	$0.99999999999971\cdots$

Table 2: Numerical values of $\gamma(\beta)$ that correspond to the first couple '$\star$' in Table 1.

Theorem 4.1. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ with $a, b > 0$ and $\frac{a}{b} \in \mathbb{Z}$. We have that $\gamma(\beta) = 1$ if and only if $a \geq b^2$ or $(a, b) \in \{(6, 24), (6, 30)\}$.*
If $a = b \geq 3$ then $\gamma(\beta) = 0$.
If $b \leq a \leq b(b-1)$ then $\gamma(\beta)$ can be computed with arbitrary precision.

The two cases $\beta^2 = 24\beta + 6$ and $\beta^2 = 30\beta + 6$ are very exceptional. It is given by the fact that for them, we have that $b - (a/b)$ divides b, which is an important ingredient in their strangeness. Table 1 shows whether $\gamma(\beta)$ is 0, 1 or strictly in between, for $b \leq 12$ and $a/b \leq 15$. The first non-trivial values are listed in Table 2.

Example 4.2. As an example, we will show the computation of $\gamma(\beta)$ for $\beta = 1 + \sqrt{3}$, the Pisot root of $\beta^2 = 2\beta + 2$. Since b divides a, we know that we can choose every odd digit and the even digit is then given uniquely. This allows us to consider shorter intervals than the ones in Lemma 3.4, namely, $[1 + P'(w) + (\beta')^{2n+1}\frac{b-1}{1-(\beta')^2}, 1 + P'(w)]$ for a prefix w of the length $2n$.

The computation is shown in Figure 2. We start with the interval for the empty word, which is $\left[1 - \frac{\beta'(b-1)}{1-(\beta')^2}, 1\right]$. We then take the two values 0, 1 for the 1st digit; the second digit is fixed by this and we get the two prefixes 00 and 10. However, the interval for 10 does not overlap the left-most interval (the one for 00 in this case), therefore we can 'forget' it. In each step, we then extend the length of the prefixes by two and we 'forget' the intervals that do not overlap the left-most one. The value of $\gamma(\beta)$ lies in the left-most interval. Already in the 5th step we obtain that $\gamma(\beta) \in [0.922, 0.971]$ therefore it is strictly between 0 and 1.

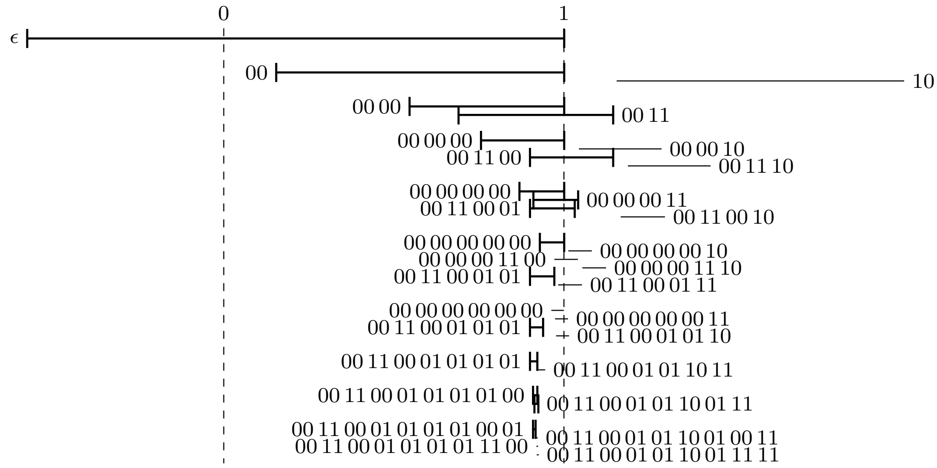

Figure 2: The computation of $\gamma(1+\sqrt{3})$. By a thick line we denote the intervals that we keep, by a thin line the ones that we 'forget'.

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