Weak matching rules for quasicrystals*

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Planar tilings with n-fold rotational symmetry (*i.e.*, invariant by a rotation of angle $2\pi/n$, see, *e.g.*, Fig. 1) are commonly used to model the long range aperiodic order of quasicrystals. In this context, it is important to know which tilings are characterized only by *local rules*. Local rules are constraints on the way neighboor tiles can fit together. They aim to model finite-range energetic interactions which stabilize quasicrystals. They are said to be weak if they moreover allow the tilings to have small variations which do not affect the long range order.

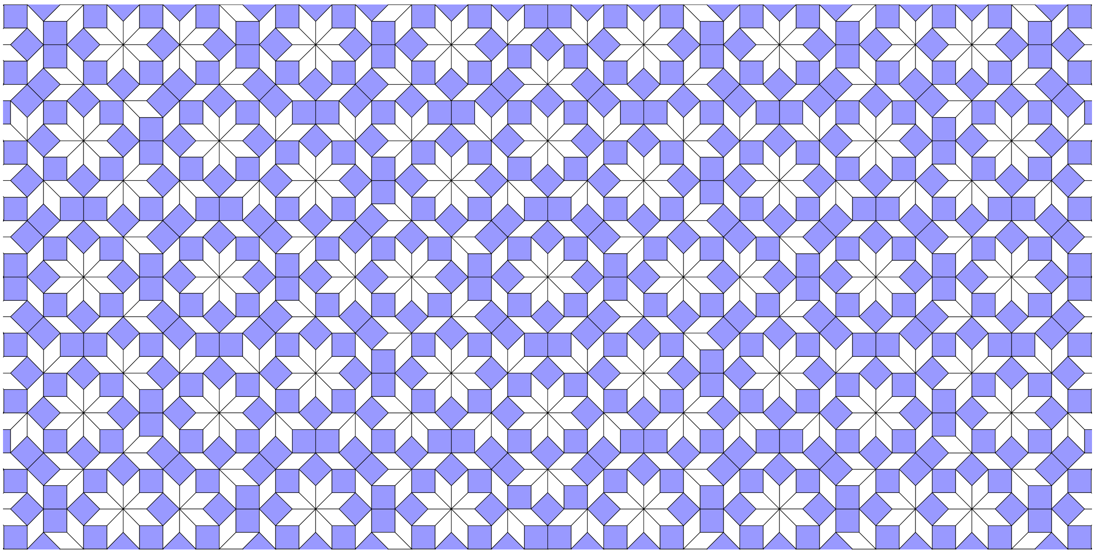

Figure 1: Patch of a planar tiling with 8-fold rotational symmetry.

On the one hand, Socolar [6] showed in 1990 that the n-fold planar tilings do admit weak local rules when n is not divisible by 4 (the $n = 5$ case corresponds to the Penrose tilings and is known since 1974). On the other hand, Burkov [3] showed in 1988 that this is not the case for $n = 8$ and Le [5] showed the same for $n = 12$. Our contribution fills the gap:

Theorem 1 *Planar tilings with a $4p$-fold rotational symmetry do not admit weak local rules.*

We will see planar tilings with n-fold rotational symmetry as digitizations of two-dimensional planes in higher dimensional spaces. The algebraic properties of these planes, in particular the rational dependencies between their Grassman coordinates, will allow us to use techniques developped by the authors in [1, 2] to exhibit for any n divisible by 4 a one-parameter family of planes that will play a crucial role. Namely, we will rely on results obtained by Julien in [4] concering the *window* associated with a planar tiling (see Fig. 2) to show that the closer a plane of this one-parameter family is of the plane corresponding to the n-fold tiling, the larger is the smallest patch that allows to distinguish between the corresponding tilings.

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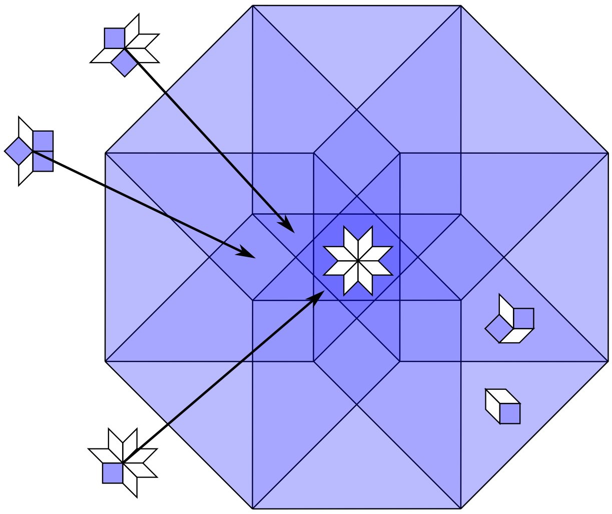

Figure 2: The window associated to the tiling of Fig. 1 depicts the ways tiles of this tiling can fit around a vertex.

References

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