Abelian properties of words associated with Parry numbers

Ondřej Turek[*]

June 2014

Abstract

Abelian complexity of a word $\mathbf{u}$ is a function that counts the number of pairwise non-abelian-equivalent factors of $\mathbf{u}$ of length n. We prove that for any c-balanced Parry word $\mathbf{u}$, the values of the abelian complexity function can be computed by a finite-state automaton. The proof is based on the notion of relative Parikh vectors. The approach works generally for any function $F(n)$ that can be expressed in terms of the set of relative Parikh vectors corresponding to the length n.

1 Introduction

Abelian complexity of a word $\mathbf{u}$ is a function $\rho_{\mathbf{u}}^{\mathrm{ab}}:\mathbb{N}\to\mathbb{N}$ that counts the number of pairwise non-abelian-equivalent factors of $\mathbf{u}$ of length n [1]. Although the notion is simple, the evaluation of $\rho_{\mathbf{u}}^{\mathrm{ab}}(n)$ for a given infinite word $\mathbf{u}$ is usually a complicated task. One possible approach to the problem consists in deriving an explicit formula for the abelian complexity function. For example, one can show that every Sturmian word satisfies $\rho_{\mathbf{u}}^{\mathrm{ab}}(n)=2$ for all $n\in\mathbb{N}$ [2]. Nevertheless, other nontrivial infinite words with an explicit formula or recurrent relations for $\rho_{\mathbf{u}}^{\mathrm{ab}}(n)$ are quite rare [1, 3, 4, 5, 6]. Moreover, achieved results are related only to words over binary and ternary alphabets.

Another approach consists in calculating values $\rho_{\mathbf{u}}^{\mathrm{ab}}(n)$ from definition. That is, one slides a window of size n on a sufficiently long prefix of $\mathbf{u}$ and counts the classes of abelian-equivalent factors. Despite this way is straightforward and universal, it is a brute-force method that can be used in practice only for small values of n. The length of the prefix that must be sought through is typically much greater than n, thus the calculation for large n becomes extremely slow, and even when a powerful computer is used, it sooner or later fails for memory reasons.

We are going to deal with an approach that is, in a way, a combination of the previous two ones. We show that for any c-balanced Parry word $\mathbf{u}$, values $\rho_{\mathbf{u}}^{\mathrm{ab}}(n)$ can be calculated by a finite-state automaton with a normal U-representation of n as its input. In other words, instead of sliding a window of size n on a certain prefix of $\mathbf{u}$, which is inconvenient because the required prefix length grows to infinity as $n\to\infty$, it gives one the possibility to perform a walk on a transition diagram of a discrete finite-state automaton, which is a *finite* graph, independent of n. The result can be iterpreted also in the way that there exist functions δ and τ allowing to evaluate $\rho_{\mathbf{u}}^{\mathrm{ab}}(n)$ in $\mathcal{O}(\log n)$ steps. Our proof is constructive; we show how to derive the finite-state automaton in question for a given word $\mathbf{u}$, i.e., we explain how to find the functions δ and τ.

[*]Corresponding author, Nuclear Physics Institute, Academy of Sciences of the Czech Republic, Řež, Czech Republic *and* Bogolyubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia

2 Preliminaries

Let us consider an infinite word $\mathbf{u}$ over the alphabet $\mathcal{A} = \{0, 1, \ldots, m-1\}$. For every finite factor w of $\mathbf{u}$, we define the *Parikh vector* of w as the m-tuple $\Psi(w) = (|w|_0, |w|_1, \ldots, |w|_{m-1})$, where $|w|_\ell$ for $\ell \in \mathcal{A}$ denotes the number of occurences of the letter ℓ in w. If we denote the length of w by $|w|$, it obviously holds $|w|_0 + |w|_1 + \cdots + |w|_{m-1} = |w|$. Let us define

$$\mathcal{P}_{\mathbf{u}}(n) = \{\Psi(w)\,;\, w \text{ is a factor of } \mathbf{u}, |w| = n\}\,.$$

The *abelian complexity* of a word $\mathbf{u}$ is the function $\rho^{\text{ab}}_{\mathbf{u}} : \mathbb{N} \to \mathbb{N}$ counting the elements of $\mathcal{P}_{\mathbf{u}}(n)$,

$$\rho^{\text{ab}}_{\mathbf{u}}(n) = \#\mathcal{P}_{\mathbf{u}}(n)\,, \tag{1}$$

where # denotes the cardinality.

The *relative Parikh vector* [7] is defined for any factor w of $\mathbf{u}$ of length n as

$$\Psi^{\text{rel}}_{\mathbf{u}}(w) = \Psi(w) - \Psi(u_0 u_1 \cdots u_{n-1})\,, \tag{2}$$

where $u_0 u_1 \cdots u_{n-1}$ is the prefix of $\mathbf{u}$ of length n. Since the subtrahend $\Psi(u_0 u_1 \cdots u_{n-1})$ on the right-hand side of (2) does not depend on w, the set of relative Parikh vectors corresponding to the length n,

$$\mathcal{P}^{\text{rel}}_{\mathbf{u}}(n) := \left\{\Psi^{\text{rel}}_{\mathbf{u}}(w)\ ;\ w \text{ is a factor of } \mathbf{u}, |w| = n\right\},$$

has the same cardinality as $\mathcal{P}_{\mathbf{u}}(n)$. Hence we obtain, with regard to (1),

$$\rho^{\text{ab}}_{\mathbf{u}}(n) = \#\mathcal{P}^{\text{rel}}_{\mathbf{u}}(n)\,. \tag{3}$$

Parry words are infinite words associated with the set of β-integers for Parry numbers β. A *simple Parry word* over the alphabet $\mathcal{A} = \{0, 1, \ldots, m-1\}$ is a fixed point of a substitution

$$\begin{array}{lrcl} \varphi: & 0 & \mapsto & 0^{\alpha_0}1 \\ & 1 & \mapsto & 0^{\alpha_1}2 \\ & & \vdots & \\ & m-2 & \mapsto & 0^{\alpha_{m-2}}(m-1) \\ & m-1 & \mapsto & 0^{\alpha_{m-1}} \end{array} \tag{4}$$

A *non-simple Parry word* over the alphabet $\mathcal{A} = \{0, 1, \ldots, m+p-1\}$ is a fixed point of

$$\begin{array}{lrcl} \varphi: & 0 & \mapsto & 0^{\alpha_0}1 \\ & 1 & \mapsto & 0^{\alpha_1}2 \\ & & \vdots & \\ & m+p-2 & \mapsto & 0^{\alpha_{m+p-2}}(m+p-1) \\ & m+p-1 & \mapsto & 0^{\alpha_{m+p-1}}m \end{array} \tag{5}$$

The exponents α_j occurring in (4) and (5) are non-negative integers obeying certain restrictions [8, 9]. Both substitutions must satisfy $\alpha_0 \geq 1$ and $\alpha_\ell \leq \alpha_0$ for all $\ell \in \mathcal{A}$. In addition, (4) requires $\alpha_{m-1} \geq 1$, whereas (5) requires $\alpha_\ell \geq 1$ for a certain $\ell \in \{m, m+1, \ldots, m+p-1\}$.

For a given substitution (4) or (5), let us set $U_j = |\varphi^j(0)|$ for every $j \in \mathbb{N}_0$. Any $n \in \mathbb{N}$ can be represented as a sum $n = \sum_{j=0}^{k} d_j U_j$ with $d_j \in \mathbb{N}_0$. If coefficients d_j are obtained by the greedy algorithm, the sequence $d_k d_{k-1} \cdots d_1 d_0$ is called *normal U-representation* of n [10] and denoted

$$\langle n \rangle_U = d_k d_{k-1} \cdots d_1 d_0\,.$$

The coefficients obtained by the greedy algorithm satisfy $d_j \in \{0,1,\ldots,\alpha_0\}$ for all $j = 0,1,\ldots,k$. The *incidence matrix* $\mathcal{M}_\varphi$ of a substitution φ on $\mathcal{A} = \{0,1,\ldots,m-1\}$ is defined by

$$\mathcal{M}_\varphi = \begin{pmatrix} |\varphi(0)|_0 & |\varphi(0)|_1 & \cdots & |\varphi(0)|_{m-1} \\ \vdots & \vdots & & \vdots \\ |\varphi(m-1)|_0 & |\varphi(m-1)|_1 & \cdots & |\varphi(m-1)|_{m-1} \end{pmatrix} .$$

It follows immediately from the definition of $\mathcal{M}_\varphi$ that for any $w \in \mathcal{A}^*$,

$$\Psi(\varphi(w)) = \Psi(w)\mathcal{M}_\varphi .$$

Furthermore, due to [11], if all the eigenvalues of $\mathcal{M}_\varphi$ except the dominant one are of modulus less than one, then the fixed point of φ is c-balanced for a certain c, i.e., for every $\ell \in \mathcal{A}$ and for every pair of factors v, w of $\mathbf{u}$ such that $|v| = |w|$, it holds $||v|_\ell - |w|_\ell| \leq c$.

A sequence $(a_n)_{n\in\mathbb{N}}$ with values in a finite alphabet Δ is called U*-automatic* (cf. [12]) if there exists a deterministic finite automaton with output, $(Q, \Sigma, \delta, q_0, \Delta, \tau)$, with the input alphabet $\Sigma = \{0,1,\ldots,\alpha_0\}$, a transition function δ, an initial state q_0 and an output function τ such that

$$a_n = \tau(\delta(q_0, \langle n\rangle_U)) \qquad \text{for all } n \in \mathbb{N}.$$

Here we assume that the domain of δ is extended to $Q \times \Sigma^*$ by defining $\delta(q,\epsilon) = q$ for all states $q \in Q$ and $\delta(q, xa) = \delta(\delta(q,x), a)$ for all $q \in Q$, $x \in \Sigma^*$ and $a \in \Sigma$, cf. [13].

3 Abelian complexity of c-balanced Parry words

From now on let $\mathbf{u}$ be a Parry word, i.e., the fixed point of a substitution (4) or (5). In addition, we assume that $\mathbf{u}$ is c-balanced for a certain $c > 0$. We aim to prove that under these assumptions, the sequence $\left(\rho^{\mathrm{ab}}_{\mathbf{u}}(n)\right)_{n=1}^{\infty}$ is U-automatic.

3.1 The main idea

Our strategy consists in introducing certain finite sets $\mathcal{S}(n)$ for $n \in \mathbb{N}$ (their structure will be described below) with the following properties.

(P1) For any $n \in \mathbb{N}$, the set of relative Parikh vectors $\mathcal{P}^{\mathrm{rel}}_{\mathbf{u}}(n)$ can be constructed from $\mathcal{S}(n)$.

(P2) There exists a finite number of sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_M$ such that for any $n \in \mathbb{N}$, $\mathcal{S}(n) = \mathcal{S}_j$ for a certain $j \in \{1,2,\ldots,M\}$.

(P3) If the normal U-representation of a number $N \in \mathbb{N}$ satisfies $\langle N\rangle_U = \langle n\rangle_U d$ for certain $n \in \mathbb{N}$ and $d \in \{0,1,\ldots,\alpha_0\}$, then the set $\mathcal{S}(N)$ can be constructed from $\mathcal{S}(n)$.

Property (P2) combined with property (P1) guarantees the existence of finitely many sets of relative Parikh vectors, $\mathcal{P}^{\mathrm{rel}}_1, \ldots, \mathcal{P}^{\mathrm{rel}}_M$, such that $\mathcal{S}(n) = \mathcal{S}_j \Rightarrow \mathcal{P}^{\mathrm{rel}}_{\mathbf{u}}(n) = \mathcal{P}^{\mathrm{rel}}_j$. At the same time, combining property (P2) with property (P3) allows us to define a function $\delta(j,d)$ such that $\left(\, \mathcal{S}(n) = \mathcal{S}_j \wedge \langle N\rangle_U = \langle n\rangle_U d \,\right) \Rightarrow \mathcal{S}(N) = \mathcal{S}_{\delta(j,d)}$.

Once the sets $\mathcal{P}^{\mathrm{rel}}_1, \ldots, \mathcal{P}^{\mathrm{rel}}_M$ are established, one can introduce a function $\tau : \{1,2,\ldots,M\} \to \mathbb{N}$ defined as $\tau(j) = \#\mathcal{P}^{\mathrm{rel}}_j$. Then the calculation of $\rho^{\mathrm{ab}}_{\mathbf{u}}(n)$ for a given $n \in \mathbb{N}$ is carried out as follows. In the first step, the function δ is used to tranform $\langle n\rangle_U$ into the value j such that $\mathcal{S}(n) = \mathcal{S}_j$. Note that j can attain only values $1,\ldots,M$, thus a machine with *finitely many states* is sufficient to perform the procedure. In the second step, the function τ is used to transform the value j into the value $\rho^{\mathrm{ab}}_{\mathbf{u}}(n)$. It holds $\mathcal{S}(n) = \mathcal{S}_j \;\Rightarrow\; \mathcal{P}^{\mathrm{rel}}_{\mathbf{u}}(n) = \mathcal{P}_j \;\Rightarrow\; \rho^{\mathrm{ab}}_{\mathbf{u}}(n) = \tau(j)$, cf. equation (3).

3.2 Definition of $\mathcal{S}(n)$

For any finite factor w of $\mathbf{u}$, let h_w be the sum of components of the vector $\Psi^{\mathrm{rel}}_{\mathbf{u}}(w)\mathcal{M}_\varphi$. Since $\mathbf{u}$ is c-balanced by assumption, the set $\{\Psi^{\mathrm{rel}}_{\mathbf{u}}(w)\,;\, w \text{ is a factor of } \mathbf{u}\}$ is finite, hence the set $\{|h_w|\,;\, w \text{ is a factor of } \mathbf{u}\}$ is finite as well. We put H to be any (fixed) number satisfying the inequality $H \geq \max\{|h_w|\,;\, w \text{ is a factor of } \mathbf{u}\}$. Furthermore, let L be any (fixed) number such that the implication

$$|w| \geq L \quad \Rightarrow \quad |\varphi(w)| - |w| \geq 2\alpha_0 + H \tag{6}$$

holds true for all factors w of $\mathbf{u}$. The existence of L follows from the recurrence of $\mathbf{u}$.

Definition 3.1. Let L be the number introduced by equation (6). For all $n \in \mathbb{N}$, we define the set

$$\mathcal{S}(n) = \left\{ \left(\Psi^{\mathrm{rel}}_{\mathbf{u}}(u_j u_{j+1} \cdots u_{j+n-1}), u_j, u_{j+n-L} \cdots u_{j+n+L} \right) \;;\; j \geq L \right\}. \tag{7}$$

The set $\mathcal{S}(n)$ consists of triples $(\psi, a, b_{-L} \cdots b_0 \cdots b_L)$, where

- $\psi = \Psi^{\mathrm{rel}}_{\mathbf{u}}(w)$ is the relative Parikh vector of a certain factor w of $\mathbf{u}$ of length n;
- $a \in \mathcal{A}$ is the first letter of w;
- $b_{-L} \cdots b_L$ is a factor of $\mathbf{u}$ of length $2L+1$; its middle letter b_0 coincides with the successor of the last letter of w in $\mathbf{u}$.

The relative positions of w, a and $b_{-L} \cdots b_0 \cdots b_L$ in $\mathbf{u}$ can be illustrated in the following way:

$$\mathbf{u} = u_0 \cdots u_{j-1} \underbrace{\overbrace{u_j}^{a} \, u_{j+1} \cdots \overbrace{u_{j+n-L} \cdots u_{j+n-1}}^{b_{-L}\cdots b_{-1}}}_{w} \overbrace{u_{j+n}}^{b_0} \overbrace{u_{j+n+1} \cdots u_{j+n+L}}^{b_1 \cdots b_L} u_{j+n+L+1} \cdots .$$

3.3 Property (P1)

Observation 3.2. *For all $n \in \mathbb{N}$, it holds*

$$\mathcal{P}^{\mathrm{rel}}_{\mathbf{u}}(n) = \{\psi \;;\; (\psi, a, b_{-L} \cdots b_0 \cdots b_L) \in \mathcal{S}(n)\}. \tag{8}$$

Proof. The statement follows from equation (7) and from the fact that $\mathbf{u}$ is recurrent. □

3.4 Property (P2)

Proposition 3.3. *There exist sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_M$ such that*

$$(\forall n \in \mathbb{N})\ (\exists j \in \{1, 2, \ldots, M\})\ (\mathcal{S}(n) = \mathcal{S}_j).$$

Proof. The c-balancedness of $\mathbf{u}$ implies that the union $\bigcup_{n=1}^{\infty} \mathcal{S}(n)$ contains finitely many elements. Since $\mathcal{S}(n)$ for $n \in \mathbb{N}$ are subsets of $\bigcup_{n=1}^{\infty} \mathcal{S}(n)$, there is only a finite number of them. □

3.5 Property (P3)

Proposition 3.4. *There exists an algorithm transforming the set $\mathcal{S}(n)$ into the set $\mathcal{S}(N)$ for any pair of integers $n, N \in \mathbb{N}$ such that $\langle N \rangle_U = \langle n \rangle_U d$ for a certain $d \in \{0, 1, \ldots, \alpha_0\}$, i.e., $\langle n \rangle_U = d_k \cdots d_0$, $\langle N \rangle_U = d_k \cdots d_0 d$.*

The algorithm consists in taking the elements $(\psi, a, b_{-L} \cdots b_L) \in \mathcal{S}(n)$ one by one, and in applying a formula that transforms $(\psi, a, b_{-L} \cdots b_L)$ into a certain set of triples $(\hat{\psi}, \hat{a}, \hat{b}_{-L} \cdots \hat{b}_L)$. The union of all triples $(\hat{\psi}, \hat{a}, \hat{b}_{-L} \cdots \hat{b}_L)$ constructed in this way constitutes the set $\mathcal{S}(N)$.

3.6 U-automaticity

Proposition 3.5. *There exists a function* $\delta(j,d)$ *for* $j \in \{1,\dots,M\}$ *and* $d \in \{0,\dots,\alpha_0\}$ *such that for any pair* $n, N \in \mathbb{N}$ *satisfying*

$$\langle n\rangle_U = d_k d_{k-1}\cdots d_1 d_0 \qquad \textit{and} \qquad \langle N\rangle_U = d_k d_{k-1}\cdots d_1 d_0 d$$

it holds

$$\mathcal{S}(n) = \mathcal{S}_j \qquad \Rightarrow \qquad \mathcal{S}(N) = \mathcal{S}_{\delta(j,d)}\,. \tag{9}$$

Proof. The statement is a straightforward corollary of Propositions 3.3 and 3.4. □

We may assume without loss of generality that the sets $\mathcal{S}_j$ are enumerated so that $\mathcal{S}_d = \mathcal{S}(d)$ for all $d = 1,\dots,\alpha_0$. We also extend the definition of δ to the value $j = 0$ in the way $\delta(0,d) := d$ for all $d = 0,1,\dots,\alpha_0$. These two assumption make the implication (9) valid also for pairs n, N such that $n = 0$ and $N \in \{1,\dots,\alpha_0\}$.

Proposition 3.6. *Let* $n \in \mathbb{N}$. *It holds* $\mathcal{S}(n) = \mathcal{S}_j$ *for* $j = \delta(0,\langle n\rangle_U)$.

Proof. The formula can be proven by induction on k. Recall that the symbol $\delta(0, d_k d_{k-1}\cdots d_1 d_0)$ has the meaning $\delta(\delta(\cdots\delta(\delta(0,d_k),d_{k-1})\cdots,d_1),d_0)$, cf. Section 2. □

Let us define sets $\mathcal{P}_1^{\mathrm{rel}},\dots,\mathcal{P}_M^{\mathrm{rel}}$ as follows,

$$\mathcal{P}_j^{\mathrm{rel}} = \{\psi\ ;\ (\psi, a, b_{-L}\cdots b_L) \in \mathcal{S}_j\} \qquad \text{for all } j = 1,\dots,M\,.$$

Equation (8) with Proposition 3.6 lead to the formula

$$\mathcal{P}_{\mathbf{u}}^{\mathrm{rel}}(n) = \mathcal{P}_{\delta(0,\langle n\rangle_U)}^{\mathrm{rel}}\,. \tag{10}$$

Consequently, there exists a finite number of sets of relative Parikh vectors, $\mathcal{P}_1^{\mathrm{rel}},\dots,\mathcal{P}_M^{\mathrm{rel}}$, such that for any $n \in \mathbb{N}$, $\mathcal{P}_{\mathbf{u}}^{\mathrm{rel}}(n)$ is equal to $\mathcal{P}_j^{\mathrm{rel}}$ for a certain $j \in \{1,\dots,M\}$.

Recall that the abelian complexity $\rho_{\mathbf{u}}^{\mathrm{ab}}(n)$ is equal to the cardinality of the set $\mathcal{P}_{\mathbf{u}}^{\mathrm{rel}}(n)$, cf. equation (3). With regard to that, we introduce a function $\tau : \{1,\dots,M\} \to \mathbb{N}$ by the relation

$$\tau(j) = \#\mathcal{P}_j^{\mathrm{rel}}\,. \tag{11}$$

Combining equations (3), (10) and the definition (11), we obtain the main result:

Theorem 3.7. *The abelian complexity of* $\mathbf{u}$ *is given by the formula*

$$\rho_{\mathbf{u}}^{\mathrm{ab}}(n) = \tau\left(\delta(0,\langle n\rangle_U)\right)\,. \tag{12}$$

Equation (12) implies that the sequence $\left(\rho_{\mathbf{u}}^{\mathrm{ab}}(n)\right)_{n=1}^{\infty}$ is U-automatic.

Remark 3.8. The whole argument relies on the existence of sets $\mathcal{S}_1,\dots,\mathcal{S}_M$ with properties referred to as (P1), (P2) and (P3). It can be shown that such sets $\mathcal{S}_1,\dots,\mathcal{S}_M$ can be found explicitly by a quite simple algorithm.

Remark 3.9. For any $n \in \mathbb{N}$, the set of relative Parikh vectors $\mathcal{P}_{\mathbf{u}}^{\mathrm{rel}}(n)$ is equal to $\mathcal{P}_j^{\mathrm{rel}}$, where the value j is assigned to n by a finite automaton using the transition function δ, cf. equation (10). Consequently, any function $F : \mathbb{N} \to \mathbb{N}$ that is defined in terms of the set of relative Parikh vectors $\mathcal{P}_{\mathbf{u}}^{\mathrm{rel}}(n)$ can be evaluated by a finite automaton using the transition function δ and an appropriate output function τ_F. For example, the *balance function* [14, 15] of a word $\mathbf{u}$ is defined as

$$B_{\mathbf{u}}(n) = \max\{\big|\,|w|_a - |w'|_a\big|\ ;\ a \in \mathcal{A}, w, w' \text{ are factors of } \mathbf{u},\ |w| = |w'| = n\}\,.$$

It is easy to show that the right hand side is equal to $\max\{\|\psi - \psi'\|_\infty ; \psi, \psi' \in \mathcal{P}^{\mathrm{rel}}_{\mathbf{u}}(n)\}$. Therefore, if we define the output function $\tau_B(j) := \max\left\{\|\psi - \psi'\|_\infty ; \psi, \psi' \in \mathcal{P}^{\mathrm{rel}}_j\right\}$ for all $j = 1, \dots, M$, we can write

$$B_{\mathbf{u}}(n) = \tau_B\left(\delta(0, \langle n\rangle_U)\right) .$$

Hence, the sequence $(B_{\mathbf{u}}(n))_{n=1}^{\infty}$ is U-automatic. A similar result can be obtained for any other function expressible in terms of the set of relative Parikh vectors $\mathcal{P}^{\mathrm{rel}}_{\mathbf{u}}(n)$.

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