Abstract

In this talk we will present a study of the equation \( [x] = [x + a] + s \) where \( x \) and \( a \) are positive integers, \( s \) is an integer and \( [x] \) denotes the number of “1” in the binary decomposition of \( x \). We will be interested in solving this equation for fixed \( a \) and \( s \) as well as the statistical behaviour of \( [x] - [x + a] \) for a fixed positive integer \( a \).

1 Introduction

Let \( x \) be a positive integer and \( [x] \) denote the number of “1” in the binary expansion of \( x \). We are interested in solving the equation \( [x] = [x + a] + s \) for fixed \( a \in \mathbb{N} \) and \( s \in \mathbb{Z} \). The means employed for such a problem are mainly combinatorial via the construction of a summation tree. Knowing the structure of solutions of such an equations allows, for each \( a \in \mathbb{N} \), the study of the distribution of probability of the difference \( [x] - [x + a] \), given by the function \( \mu_a \) over \( l^1(\mathbb{Z}) \), where \( x \) can be identified with its binary expansion and so as a sequence of 0 and 1 with balanced Bernouilli distribution of probability. To this end, we study further the summation tree we introduced earlier. Being able to compute such a probability measure for each positive integer \( a \) we then focus on the study of its asymptotic behaviour as \( a \) gets large. This involves looking at pathes in a particular Schreier graph of the Baumslag-Solitar group of type \( (1, 2) \).

2 Results

We wish to have a precise, constructive, understanding of the solutions of the equation \( [x] = [x + a] + s \) for any set of parameters \( a \) and \( s \). To this end, we construct an infinite binary tree associated to \( a \) on which it is possible to read the binary expansion of solutions to this equation as pathes on this tree. An example of a part of such a tree is given on figure 1. Such a construction allows us to prove the following theorem:

**Theorem 1** Let \( a \in \mathbb{N} \) and \( s \in \mathbb{Z} \). There exists a finite set of prefixes

\[
\mathcal{P} = \{p_1, ..., p_k\} \subset \{0, 1\}^*
\]

such that \( x \in \mathbb{N} \) is solution of \( [x] = [x + a] + s \) if and only if the binary expansion of \( x \) starts with one of the prefixes \( p_i \).

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Let us now define, for any positive integer \( a \), the function \( \mu_a \in l^1(\mathbb{Z}) \) defined by

\[
\forall s \in \mathbb{Z}, \quad \mu_a(s) = \mathbb{P}(\{ x \in \mathbb{N} \mid [x] - [x + a] = s \})
\]

where \( \mathbb{P} \) is the balanced Bernoulli probability measure on \( \{0, 1\}^\ast \) and by identifying the integer \( x \) and its binary expansion. Collapsing the tree on a particular Bratelli diagram as shown in figure 2 and understanding its patterns allows us to prove the next theorem:

**Theorem 2** The function \( \mu_a \) is calculated via an infinite product of matrices

\[
\mu_a = (1, 1) \cdots A_{a_n} A_{a_{n-1}} \cdots A_{a_1} A_0 \left( \begin{array}{c} \delta_0 \\ 0 \end{array} \right),
\]

where \( a = a_0 + 2a_1 + \ldots \) is a binary expansion of \( a \), \( \delta_0(i) = \begin{cases} 0 & \text{if } i \neq 0 \\ 1 & \text{otherwise} \end{cases} \)

\[
A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \hat{\sigma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{1}{2} \hat{\sigma} & 0 \\ \frac{1}{2} \hat{\sigma}^{-1} & 1 \end{pmatrix},
\]

and \( \hat{\sigma} : (p_j) \mapsto (p_{j+1}) \) is the shift transformation on \( l^1(\mathbb{Z}) \).
The final part of our investigation is dedicated to the asymptotic behaviour of $\mu_a$ for large integers $a$. We have to use the following object:

**Definition 1** Let $G$ be a finitely generated group with a generator set $S$, and let $H$ be a subgroup, not necessary normal, such that $S \cap H = \emptyset$. The Schreier graph for the triple $(G, H, S)$ is defined as the orientated graph with vertex set $G/H$ and edge set $E = \{(aH, saH) | a \in G, s \in S\}$.

The group we wish to consider is the Baumslag-Solitar group of type $(1,2)$ which is defined by

$$BS(1, 2) = \langle \sigma, S \mid \sigma S \sigma^{-1} = S^0 \rangle$$

in the particular case where the generators are the following real functions

$$\sigma : y \to 2y, \quad S : y \to y + 1.$$ 

This group naturally acts on the set of diadic integers. Then, for the generator set $\{S, S^{-1}, \sigma\}$ and a certain subgroup of $BS(1, 2)$, there is an associated Schreier graph $\Gamma$ where it is possible to associate vertices to diadic integers.

Then, for all integer $a$, denote by $\gamma_a$ the geodesic linking 0 to $a$ in $\Gamma$ and denote by $w$ the weight function on $BS(1, 2)$ taking value 1 on $S, S^{-1}$ and 0 on $\sigma, \sigma^{-1}$. Finally, let $\|a\|_0 = \int_{\gamma_a} w(g) \, dg$. We have the following result:

**Theorem 3** For $\|a\|_0$ large enough, we have the following inequality:

$$\|\mu_a\|_2 \leq C_0 \cdot \|a\|_0^{-1/4}$$

where $C_0$ is a universal constant.

The study of such a problem is motivated by its links with some ergodic properties of Vershik’s transformation in the Pascal triangle.