Emergence of regularities on decreasing sandpile models*

Kévin Perrot[†] and Éric Rémila[‡]

Abstract

Sandpile is a class of conservative discrete dynamical systems, where cubic sand grains move around according to local rules. One naturally wants sand heaps to be decreasing, as conical heaps formed in hourglasses. This extended abstract introduces convenient tools towards a characterization of emergent regularities in decreasing sandpile models: recurrence automata. Interestingly, the technic presented here allows to deal with very regular sand patterns emerging from an initial phase of apparent disorder, without necessitating a precise understanding of this latter. It generalizes earlier works presented in [17, 18], and asks how far can we go following this track?

1 Introduction

Interaction driven systems are everywhere in nature, and understanding their dynamic is a present challenge. Sandpile models provide a formal frame of work for the mathematical study of emergent structures. After the seminal works of Bak, Tang, Wiesenfeld [2], and later Dhar [3], their received great attention. From the basic models [7, 8, 10, 11, 14, 16, 19] to more involved ones [4, 5, 6, 12, 17, 18], a motivating goal has to to describe the shape of fixed point configurations, thus answering the question: where does the dynamic leads to?

This extended abstract concentrates on the whole class of one-dimensional sandpile models verifying only one property: decrease. Consider an hourglass, in the absence of wind it is natural to require the configuration of sand grains, in the dynamical process letting grains fall from the top and avalanching on the sides, to be decreasing so that it looks approximately like a cone. This is what decreasing sandpile models catch.

The main result we present is the construction of a finite state automaton describing fixed point configurations. This automaton recognizes a very restricted language, which words are the only one that can possibly (or rather non-impossbly) account for the shape of the fixed point, and where emergent structures can often directly be pointed out. First, we present the definition of decreasing sandpile models together with elementary background and classical structural results in Section 2. Then, Section 3 exposes the construction of an internal dynamic of fixed points, whose iterations describe fixed points. A combination of arguments from linear algebra and combinatorics will thereafter allow to prove the emergence of regularities on fixed points, and recurrence automata embedding them will finally be presented. We conclude in Section 4 with ideas on how to improve the restrictions included in recurrence automata, in order to have a very precise description of emergent structures in the shape of fixed points.

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†Universidad de Chile - DII - DIM - CMM (UMR 2807 - CNRS), 2120 Blanco Encalada, Santiago, Chile; and Université de Lyon - LIP (UMR 5668 - CNRS - ENS de Lyon - Université Lyon 1), 46 allé d'Italie 69364 Lyon Cedex 7, France. kperrot@dim.uchile.cl

‡Université de Lyon - GATE LSE (UMR 5824 - CNRS - Université Lyon 2), Site stéphanois, 6 rue Basse des Rives, 42 023 Saint-Etienne Cedex 2, France. eric.remila@univ-st-etienne.fr

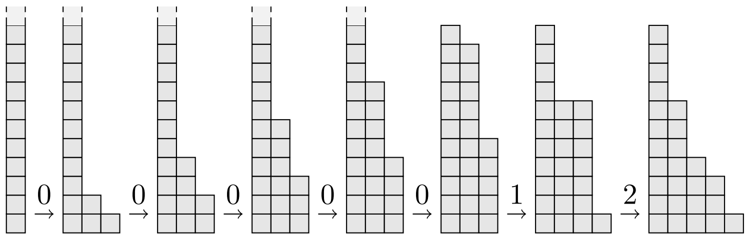

Figure 2: An example of evolution for the rule $(2,1)$, from the configuration $(26,0^\omega)$ to the fixed point $h=(11,7,4,3,1,0^\omega)$. Transitions are labeled with the fired index.

2 Decreasing sandpiles

Decreasing sandpile models are discrete dynamical systems: configurations are non-increasing sequences of integers $(h_i)_{i\in\mathbb{N}}$ where h_i is the number of stacked grains on column i; and there is a transition rule telling how grains move when the slope is too sharp. The *slope* at i is the height difference $\Delta h_i = h_i - h_{i+1}$. The rule is a p-tuple $(g_1,\ldots,g_p)$ with $g_1 \geqslant \cdots \geqslant g_p > 0$, where $G=\sum_{j=1}^{p} g_j$ grains can fall from column i if the slope Δh_i is greater or equal to $G+g_1$, then g_j of those grains land on column $i+j$ (example on Figure 1). The reason the slope should be greater or equal to $G+g_1$ is to preserve a non-increasing sequence of sand columns: G grains leave column i and g_1 land on column $i+1$, therefore Δh_i looses $G+g_1$ units of slope, and should remain non-negative.

Definition 1. *A decreasing sandpile model is a discrete dynamical system defined by the following two sets.*

- *Configurations. Infinite non-increasing integer sequences.*
- *Transition rule: A positive and non-increasing p-tuple $(g_1, g_2 \ldots, g_p)$, that is with $g_1 \geqslant \cdots \geqslant g_p > 0$. Let $G=\sum_{j=1}^{p} g_j$. From a configuration h, a transition at i leads to the configuration h', denoted $h \xrightarrow{i} h'$, such that:*
  - $h'_i = h_i - G$;
  - $h'_{i+j} = h_{i+j} + g_j$ *for* $1 \leqslant j \leqslant p$;
  - $h'_j = h_j$ *otherwise.*

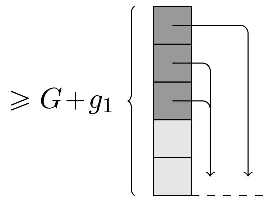

Figure 1: The rule $(2,1)$ is applied only if $\Delta h_i \geqslant 5$. *Right* is the direction of grains fall.

We also say that i is *fired*. We denote $h \to h'$ without specifying the fired column, and $\to^*$ its reflexo-transitive closure. The rule can be applied only if $\Delta h_i \geqslant G + g_1$ (G is the number of grains moving, and the total number of sand grains is conserved), *i.e.*, when i is *unstable*. We call *stable*, or a *fixed point*, a configuration that has no unstable column, and *finite* a configuration that has a finite number of sand grains. The infinite sequence of 0 is denoted 0^ω. Decreasing sandpile models are non-deterministic, the rule is applied once at each time step (example of evolution on Figure 2).

Remark 1. *The classical one-dimensional sandpile rule is the* 1*-tuple* (1)*, and the Kadanoff sandpile rule with parameter p is the p-tuple* $(1,1,\ldots,1)$.

To gain locality in the representation of configurations, that is, to gain independence on the position within the configuration, we conveniently represent them as sequences of slopes $(\Delta h_i)_{i\in\mathbb{N}}$. The fixed point of Figure 2 is for example $\Delta h = (4,3,1,2,1,0^\omega)$.

The models are non-deterministic, but every finite configuration reaches a unique fixed point.

Proposition 1. *Every finite configuration Δh converges to a unique fixed point, denoted $\pi(\Delta h)$.*

Proof sketch. Diamond property plus termination ensures confluence (see for example [1]). □

This extended abstract focuses on the fixed point reached after a finite number N of sand grains have been added on column 0. Starting from the empty configuration $\Delta h = (0^\omega)$, we add a first grain on column 0 and *stabilize* (*i.e.*, apply the rule until reaching a stable configuration), add a second grain on column 0 and stabilize, add a third grain on column 0 and stabilize, *etc*, N

times. Let $\Delta h^{\downarrow 0}$ denote configuration Δh plus one grain on column 0, and $f^{[N]}$ denote the N^{th} iteration of function f, this *inductive* procedure that adds grains one by one is therefore

$$(\pi \circ \cdot^{\downarrow 0})^{[N]}((0^\omega)) = \pi(\pi(\dots\pi((0^\omega)^{\downarrow 0})^{\downarrow 0}\dots)^{\downarrow 0}).$$

In addition, there is another *sequential* procedure, which consists in starting from the initial configuration $(N, 0^\omega)$ (we abuse notation and also denote N this configuration) and stabilizing it. The two procedures lead to the same fixed point.

Proposition 2. *(sequential definition)* $\pi(N) = (\pi \circ \cdot^{\downarrow 0})^{[N]}((0^\omega))$ *(inductive definition).*

Remark 2. $\pi(N)$ *is the sequence of* slopes Δh *of the fixed point with* N *grains.*

Proof sketch. The key fact is that $\pi(k-1)^{\downarrow 0}$ is reachable from $(k, 0^\omega)$, for any $k \in \mathbb{N}$. The result then follows by induction on $N \in \mathbb{N}$, from the unicity of the stable configuration reached. $\square$

Later on, we will construct an automata that recognizes asymptotically $\pi(N)$ (sequence of slopes), starting from a logarithmic index, that is, $(\pi(N)_i)_{i \geqslant n}$ for $n \in \mathcal{O}(\log N)$. For this result to make sense, we provide the following tight bound on the width of the fixed point.

Proposition 3. *Whatever the rule is, the width* $w(\pi(N)) = \min\{i \mid \pi(N)_j = 0 \text{ for all } j \geqslant i\}$ *of the fixed point is in* $\Theta(\sqrt{N})$*.*

Proof sketch. The fixed point $\pi(N)$ is a non-degenerated rectangular triangle of area N, so both its sides are in the order of $\sqrt{N}$. $\square$

Thus our description will asymptotically account for the whole fixed point. Let us denote by $\mathcal{C}(N)$ the set of reachable configurations from $(N, 0^\omega)$. Every configuration Δh of this set admits another representation $(v_i)_{i \in \mathbb{N}}$ called *shot vector*, where v_i is the number of times the rule has been applied on column i to go from $(N, 0^\omega)$ to Δh. For example, the shot vector of the stable configuration on Figure 2 is $v = (5, 1, 1, 0^\omega)$. The next property is straightforward to obtain.

Proposition 4. *The shot vector of each configuration in* $\mathcal{C}(N)$ *is unique.*

Following the lines of [12], one can also see that decreasing sandpile models have a lattice structure.

Theorem 1. $\mathcal{C}(N)$ *endowed with* $\rightarrow$ *has a graded lattice structure.*

We also denote Δv the sequence of differences of shot vector, such that $\Delta v_i = v_i - v_{i+1}$. Let us give an example of fixed point where regular structures emerge (all the simulations done so far exhibit similar phenomena). For $\mathcal{R} = (7, 5, 2, 1)$ and $N = 12\,345$, the fixed point $\pi(12\,345)$ is

$h =$ 570,560,543,528,520,500,487,477,457,436,434,417,409,391,382,364,355,337,328,310,301,283,274,256,247,229,220,202,193,175,166,148,139,121,112,94,85,82,66,58,40,31,13,4,1,0^ω

$\Delta h =$ 10, 17, 15, 8, 20, 13, 10, 20, 21, 2, 17, 8, 18, 9, 18, 9, 18, 9, 18, 9, 18, 9, 18, 9, 18, 9, 18, 9, 18, 9, 18, 9, 18, 9, 18, 9, 3,16, 8,18, 9,18, 9,3,1,0^ω

$v =$ 785,329,379,356,354,323,309,291,275,259,243,228,213,199,185,172,159,147,135,124,113,103, 93, 84, 75, 67, 59, 52, 45, 39, 33, 28, 23, 19, 15,12, 9, 6, 4, 2, 1,0^ω

$\Delta v =$ 456,-50, 23, 2, 31, 14, 18, 16, 16, 16, 15, 15, 14, 14, 13, 13, 12, 12, 11, 11, 10, 10, 9, 9, 8, 8, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 3, 2, 2, 1, 1,0^ω

where we see very regular patterns emerging from a (short) disordered transient sequence. We assimilate a configuration with any of its representations. The representations h (heights), Δh (slopes) and v (shot vector) of a configuration in $\mathcal{C}(N)$ are obviously linked. In particular, let $\Delta g_j = g_j - g_{j+1}$ and $\Delta g_p = g_p$, for all $i \geqslant p$ we have

$$\Delta h_i = \Delta g_p\, v_{i-p} + \dots + \Delta g_2\, v_{i-2} + \Delta g_1\, v_{i-1} - (G + g_1)\, v_i + G\, v_{i+1} \qquad (1)$$

because the slope at i was initially null and is increased by Δg_p units each time v_{i-p} is fired, increased by Δg_{p-1} units each time v_{i-p+1} is fired, *etc.*

3 Fixed points and recurrence automata

From now on, we only deal with fixed points. This section aims at defining *recurrence automata* recognizing $(\Delta h_i)_{i \geqslant n}$ for some n in $\mathcal{O}(\log N)$, which asymptotically accounts for the whole fixed point (recall Proposition 3). We first rephrase Equation (1) to get a new discrete dynamical system whose iterations describe the fixed point from left to right. This *perturbed weighted mean system* will then be expressed in matricial form, in order to prove its exponentially quick convergence to particular sequences. This means that starting from an index n logarithmic in the number of grains N, the possibilities for $(\Delta h_i)_{i \geqslant n}$ are very restricted. This result will thereafter be embedded in *recurrence automata*, a convenient tool to continue this study of emergent structures in decreasing sandpiles.

Let us begin with the construction of an internal dynamic of fixed points describing a configuration from left to right. Through simple manipulations, Equation 1 is equivalent to the recurrence relation

$$\Delta v_i = \frac{1}{G}\Big(g_1\,\Delta v_{i-1} + g_2\,\Delta v_{i-2} + \cdots + g_p\,\Delta v_{i-p}\Big) - \frac{\Delta h_i}{G} \tag{2}$$

of a discrete dynamical system describing the configuration from left to right: given p consecutive values $(\Delta v_j)_{i-p \leqslant j < i}$, we compute Δv_i as

- the average of $(\Delta v_j)_{i-p \leqslant j < i}$ weighted by $(g_1, \ldots, g_p)$ (recall that $\sum g_i = G$);
- minus a perturbation $\frac{\Delta h_i}{G}$ in order to remain in $\mathbb{Z}$ (because Δv_i is an integer).

We express Recurrence equation (2) in matricial form,

$$\Delta V_i = M\,\Delta V_{i-1} - \frac{\Delta h_i}{G}\,K \quad \text{with} \quad \Delta V_i = \begin{pmatrix} \Delta v_{i-p+1} \\ \vdots \\ \Delta v_{i-1} \\ \Delta v_i \end{pmatrix} \quad M = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ \frac{g_p}{G} & \frac{g_{p-1}}{G} & \ldots & \frac{g_1}{G} \end{pmatrix} \quad K = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

We call it the *perturbed weighted mean system*, from $\mathbb{Z}^p$ to $\mathbb{Z}^p$, which is composed of two parts.

- A linear map $M : \mathbb{R}^p \to \mathbb{R}^p$, that:
  - shifts all the values one row upward;
  - for the last component, computes the weighted mean of ΔV_{i-1}, denoted m_{i-1}.
- A perturbation subtracted to the last component, so that the result lies in $\mathbb{Z}^p$.

We take $\Delta V_{-1} = {}^t(\frac{N}{g_p}, 0, \ldots, 0, -v_0)$ as a starting point of the system. With it, Equation (3) now holds for all $i \in \mathbb{N}$. It remains fuzzy, but only a little, for v_0 can easily be bounded by

$$\frac{N}{G + g_1} - 1 \leqslant v_0 \leqslant \frac{N}{G}. \tag{4}$$

Regarding the perturbation, we have a relation for stability (left) and one for integrity (right):

$$0 \leqslant \frac{\Delta h_i}{G} < 1 + \frac{g_1}{G} \leqslant 2 \qquad \text{and} \qquad \Delta h_{i+1} \equiv G\,m_i \mod G. \tag{5}$$

Remark 3. *Importantly, at most two values of Δh_i are possible, and only one if $m_i \geqslant G + g_1$.*

Let us now study the convergence of the perturbed weighted mean system. We know quite precisely where we start (Equation (4)), and quite precisely where each iteration leads (Remark 3). We are going to prove that the system converges exponentially quickly (*i.e.*, starting from a logarithmic index) to almost uniform vectors (vectors with values very close to each other).

Notation 1. *$m_i = \frac{1}{G}(g_p, \ldots, g_1)\,\Delta V_i$ denotes the mean of ΔV_i weighted by $(g_p, \ldots, g_1)$, and $\underline{m}_i$ (resp. $\overline{m}_i$) denotes the minimal (resp. maximal) value of ΔV_i.*

The first result states an exponentially quick convergence to vectors of bounded amplitude.

Lemma 1. *There exists a constant α and a n_0 in $\mathcal{O}(\log N)$ such that $\overline{m}_{n_0} - \underline{m}_{n_0} < \alpha$.*

Proof sketch. Equation (4) implies that $\overline{m}_{-1} - \underline{m}_{-1}$ is in $\Theta(N)$. In order to prove that iterations of the perturbed weighted mean system tend to uniform vector, that is, vectors where each value is close to the mean value, we take $M_i = {}^t(m_i, \ldots, m_i) \in \mathbb{R}^p$ and study the sequence $(Z_i)_{i \in \mathbb{N}}$ where $Z_i = \Delta V_i - M_i$, which converges to $\mathbb{0}$. From Equation (3) we get a relation of the form

$$Z_i = O\,Z_{i-1} - \frac{\Delta h_i}{G}\,L$$

where O is a contracting map: its spectral radius is strictly smaller than 1 (proved with a classical result due to Eneström and Kakeya, see for example [9]). As a consequence, we can isolate the contracting map and the perturbations,

$$Z_n = O^{n+1}\, Z_{-1} + \frac{1}{G} \sum_{i=0}^{n} \Delta h_i\, O^{n-i}\, L.$$

The left part of the sum tends exponentially to $\mathbb{0}$ (see for example [13] for a discussion on contracting maps), and the right part is upper bounded by some constant $\alpha - 1$. Since the norm of Z_i is in the order of $\overline{m}_i - \underline{m}_i$, the norm of Z_{-1} is in $\Theta(N)$ and there exists an iteration n_0 in $\mathcal{O}(\log N)$ such that the left term is strictly smaller that 1, giving the result. □

Then, we have a convergence, linear in the amplitude $\overline{m}_i - \underline{m}_i$, to a sequence which is decreasing and where two consecutive values are equal or differ by one.

Lemma 2. *There exists d in $\mathcal{O}(\overline{m}_i - \underline{m}_i)$, such that for all $k \geqslant i+d$, we have $\Delta v_k - \Delta v_{k+1} \in \{0, 1\}$. Moreover, $\Delta v_{k+1} = \Delta v_k - 1$ only if $m_i - \underline{m}_i < \frac{g_1}{G}$.*

Proof sketch. This proof strongly relies on the fact that, when $\overline{m}_i \neq \underline{m}_i$, the weighted mean m_i is strictly between $\overline{m}_i$ and $\underline{m}_i$. With the perturbation, bounded by Equation (5), subtracted to it, the integer Δv_{i+1} it leads to is strictly below $\overline{m}_i$, and greater or equal to $\underline{m}_i - 1$. When the sequence is decreasing, we have $\Delta v_i = \underline{m}_i$ and it enforces that $\Delta v_i - \Delta v_{i+1} \in \{0, 1\}$.

In order to prove that the sequence is decreasing, we can first notice that while Δv_{i+1} is above or equal to $\underline{m}_i$, it is still strictly below $\overline{m}_i$ so ΔV_i tend linearly to uniform vectors. When it happens that $\Delta v_{i+1} = \underline{m}_i - 1$, the values embedded in ΔV_i must already be very close to each other, in order for the mean m_i to be just a little bit above $\underline{m}_i$, so that the perturbation, bounded by Equation (5), subtracted to it can lead to a Δv_{i+1} strictly below $\underline{m}_i$. In this case, a careful look at the constraints on the closeness of the values embedded in ΔV_i allows to conclude. □

As Lemma 2 talks about differences of differences of shot vector, we define the second derivative $(\Delta^2 v_i)_{i \in \mathbb{N}}$ with $\Delta^2 v_i = \Delta v_i - \Delta v_{i+1}$, and $\Delta^2 V_i = {}^t(\Delta^2 v_{i-p+1}, \ldots, \Delta^2 v_{i-1})$ (note that $\Delta^2 V_i$ can be computed from ΔV_i). The difference $m_i - \underline{m}_i$ can also be computed from $\Delta^2 V_i$, via the formula

$$m(\Delta^2 V_i) = \sum_{j=1}^{p-1} \left[\Delta^2 v_{i-j} \left(\sum_{k=j+1}^{p} g_k \right) \right].$$

The combination of Lemmas 1 and 2 gives the following proposition.

Proposition 5. *There exists a column n_1 in $\mathcal{O}(\log N)$, such that*

For all $i \geqslant n_1$ we have $\Delta^2 v_i \in \{0, 1\}$.

Moreover, $\Delta^2 v_i = 1$ only if $m(\Delta^2 V_i) < g_1$.

Proposition 5 gives constraints on the sequence $(\Delta^2 V_i)_{i \in \mathbb{N}}$, that are verified exponentially quickly (*i.e.*, starting form a logarithmic index). Those constraints on $(\Delta^2 V_i)_{i \geqslant n_1}$, for some n_1 in $\mathcal{O}(\log N)$, imply constraints on $(\Delta h_i)_{i \geqslant n_1 + p}$, because we can compute the latter from the former (using Equation (1) and the fact that all those representations of the fixed point end with 0^ω). Recurrence automata, which are Muller automata (a kind of Büchi automata with stronger accepting condition, see definition in Appendix A), will embed those restrictions: they will recognize only the infinite sequences of slopes that are in accordance with Proposition 5.

States of the automata will correspond to equivalence classes of vectors $\Delta^2 V_i$ according to equality. Proposition 5 states that we have asymptotically $\Delta^2 V_i \in \{0, 1\}^{p-1}$. Then, we will have a transition from one state representing a $\Delta^2 V_i$, to another state representing the $\Delta^2 V_{i+1}$, if and only if it verifies the fact that most of their values are equal (for the shift of $\Delta^2 V_i$ accounts for a large part of $\Delta^2 V_{i+1}$), and is in accordance with the second part of Proposition 5. Furthermore, the label of the transition will be the corresponding Δh_{i+1} (easily computable from $\Delta^2 V_i$ and $\Delta^2 V_{i+1}$). We do not know from which state to begin, but for the acceptance condition we know

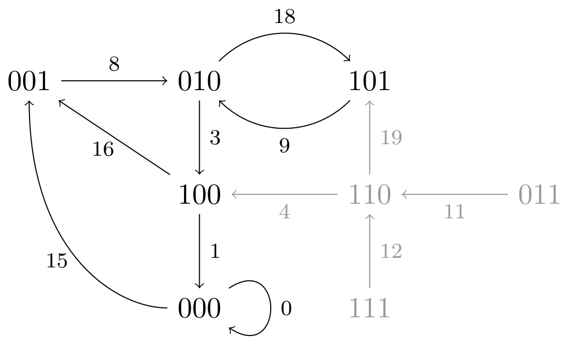

Figure 3: Recurrence automata $\mathcal{A}_{(7,5,2,1)}$. Transient states are shaded. The correspondence with $\pi(12\,456)$ (see Section 2) is:
loop between 010 and 101 .... $(18,9)^*$
do a detour via 100 and 001 .... $(3,16,8)$
loop again between 010 and 101 .... $(18,9)^*$
go to infinite loop on 000 .... $(3,1,0^\omega)$.

that the sequence $(\Delta h_i)_{i\geqslant n_1+p}$, which recurrence automata will recognize, ends with 0^ω. This always corresponds to ultimately looping forever on the state 0^{p-1}.

Definition 2. *Given a decreasing sandpile rule $\mathcal{R}$, let $\mathcal{A}_\mathcal{R}$ be its* recurrence automata*, which is the Muller automaton defined by:*

- *the set of states $\mathcal{Q}_\mathcal{R} = \{0,1\}^{p-1}$;*
- *the alphabet $\Sigma = \mathbb{N}$;*
- *the set of transitions $\rightarrow_\mathcal{R}: \mathcal{Q}_\mathcal{R} \times \Sigma \times \mathcal{Q}_\mathcal{R}$ where $q \xrightarrow{a}_\mathcal{R} q'$, with $q = q_1,\ldots,q_{p-1}$ and $q' = q'_1,\ldots,q'_{p-1}$, if and only if*
  - $(\mathcal{C}_1)$ $q'_1,\ldots,q'_{p-2} = q_2,\ldots,q_{p-1}$;
  - $(\mathcal{C}_2)$ $q'_{p-1} = 1$ *only if* $m(q) < g_1$;
  - $(\mathcal{C}_3)$ $a = m(q) + q'_{p-1}\,G$;
- *the set of initial states $\mathcal{S}_\mathcal{R} = \mathcal{Q}_\mathcal{R}$;*
- *the acceptance table $\mathcal{T}_\mathcal{R} = \langle\{0^{p-1}\}\rangle$;*

and $\mathcal{L}(\mathcal{A}_\mathcal{R})$ denotes the language of infinite words recognized by $\mathcal{A}_\mathcal{R}$.

Example of recurrence automaton on Figure 3. Theorem 2 can be considered as a convenient rephrasing of Proposition 5, and a step towards an asymptotic characterization of $(\Delta h_i)_{i\in\mathbb{N}}$.

Theorem 2. *There exists a column n_2 in $\mathcal{O}(\log N)$ such that $(\pi(N)_i)_{i\geqslant n_2} \in \mathcal{L}(\mathcal{A}_\mathcal{R})$.*

4 Conclusion and perspectives

According to Theorem 2, recurrence automata provide an asymptotically complete description of fixed points: from an index n_2 in $\mathcal{O}(\log N)$ compared to their width in $\Theta(\sqrt{N})$ (Proposition 3). Interestingly, those automata, describing emergent structures on fixed points, has been defined without requiring a fine understanding of the initial segment from columns 0 to n_2. It would be interesting to study the level of order/disorder of this part, to measure the confrontation of two ideas: it does not look simple at all, but it cannot be chaotic in a strong sense, for regularities emerge from it. Note that all the grains creating regular patterns are added on column 0 and cross this initial segment, which absolute size tends to infinity! What is precisely bypassed in this proof, and how universal is the technic in the handling of emergent structures?

Moreover, it remains to refine recurrence automata, *i.e.*, add more constraints to it in order to give a precise characterization of fixed points. We would for example like to say, according to Figure 3 and the simulation of Section 2, that rule $(7,5,2,1)$ leads to fixed points of the form $(18,9)^*\,(3,16,8)\,(18,9)^*\,(3,1,0^\omega)$. We believe that considering the inductive way of computing fixed points may now help. Indeed, from a fixed point $\pi(N)$ to the next one $\pi(N+1)$, both recognized by $\mathcal{A}_\mathcal{R}$, there are very few differences because $\pi(N+1) = \pi(\pi(N)^{\downarrow 0})$. The idea is that the runs of $\mathcal{A}_\mathcal{R}$ on $\pi(N)$ and $\pi(N+1)$ must be very close, and we think that they differ only locally. This would mean that when traveling in the automata, we are only allowed to perform very small variations, which should furthermore match at every level: of course for $(\Delta^2 v_i)_{i\geqslant n_2}$, but also for $(\Delta v_i)_{i\geqslant n_2}$ and $(v_i)_{i\geqslant n_2}$. There are actually quite few possibilities, and we are already able to issue precise characterization on some example decreasing sandpile models.

5 Aknowledgments

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References

[1] F. Baader and T. Nipkow. *Term rewriting and all that.* Cambridge University Press, New York, NY, USA, 1998.

[2] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality: An explanation of the $1/f$ noise. *Phys. Rev. Lett.*, 59:381–384, 1987.

[3] D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64:1613–1616, 1990.

[4] J. O. Durand-Lose. Parallel transient time of one-dimensional sand pile. *Theor. Comput. Sci.*, 205(1-2):183–193, 1998.

[5] E. Formenti, E. Goles, and B. Martin. Computational complexity of avalanches in the kadanoff sandpile model. *Fundam. Inform.*, 115(1):107–124, 2012.

[6] E. Formenti and B. Masson. On computing fixed points for generalized sand piles. *Int. J. on Unconventional Computing*, 2(1):13–25, 2005.

[7] E. Formenti, B. Masson, and T. Pisokas. Advances in symmetric sandpiles. *Fundam. Inform.*, 76(1-2):91–112, 2007.

[8] E. Formenti, T. Van Pham, H. D. Phan, and T. H. Tran. Fixed point forms of the parallel symmetric sandpile model. *to appear in Theor. Comput. Sci.*, 2014.

[9] R. B. Gardner and N. K. Govil. Some generalizations of the enestrÃPm-kakeya theorem. *Acta Math. Hungar.*, 74(1-2):125–134, 1997.

[10] E. Goles and M. Kiwi. Games on line graphs and sand piles. *Theor. Comput. Sci.*, 115(2):321–349, 1993.

[11] E. Goles, M. Latapy, C. Magnien, M. Morvan, and H. D. Phan. Sandpile models and lattices: a comprehensive survey. *Theor. Comput. Sci.*, 322(2):383–407, 2004.

[12] E. Goles, M. Morvan, and H. D. Phan. The structure of a linear chip firing game and related models. *Theor. Comput. Sci.*, 270(1-2):827–841, 2002.

[13] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 1996.

[14] M. Latapy, R. Mantaci, M. Morvan, and H. D. Phan. Structure of some sand piles model. *Theoretical Computer Science*, 262, 1998.

[15] D. E. Muller. Infinite sequences and finite machines. In *Proceedings of the 1963 Proceedings of the Fourth Annual Symposium on Switching Circuit Theory and Logical Design*, SWCT '63, pages 3–16, Washington, DC, USA, 1963. IEEE Computer Society.

[16] K. Perrot, H. D. Phan, and T. Van Pham. On the set of Fixed Points of the Parallel Symmetric Sand Pile Model. *Full Papers AUTOMATA 2011*, pages 17–28, 2011.

[17] K. Perrot and É Rémila. Kadanoff sand pile model. avalanche structure and wave shape. *Theor. Comput. Sci.*, 504:52–72, 2013.

[18] K. Perrot and É. Rémila. Emergence of wave patterns on kadanoff sandpiles. *LATIN*, pages 634–647, 2014.

[19] H. D. Phan. Two sided sand piles model and unimodal sequences. *ITA*, 42(3):631–646, 2008.

A Muller automata

Muller automata where introduced in [15]. They are very close to Büchi automata and make use of the concepts of *automata* and *runs*.

Definition 3. *An* automaton *is a 4-tuple* $(\mathcal{Q}, \Sigma, \to, \mathcal{S})$*, where*

- $\mathcal{Q}$ *is a set of states;*
- Σ *is an alphabel;*
- $\mathcal{S}$ *is a set of initial states;*
- $\to \subseteq \mathcal{Q} \times \Sigma \times \mathcal{Q}$ *is a set of transitions, and* $(q, a, q') \in \to$ *is denoted* $q \xrightarrow{a} q'$*.*

An automaton describes a dynamic that is captured by *runs*.

Definition 4. *For an automaton* $\mathcal{A} = (\mathcal{Q}, \Sigma, \to, \mathcal{S})$ *and an infinite input word* $w \in \Sigma^\omega$*, a* run *of* $\mathcal{A}$ *on* w *is an infinite sequence of states* $\rho \in \mathcal{Q}^\omega$ *starting at some* $\rho_0 \in \mathcal{S}$ *and such that for all* $i \in \mathbb{N}$ *we have* $\rho_i \xrightarrow{w_0} \rho_{i+1}$*.*

The definition of Muller automata adds an accepting condition for the recognition of infinite words.

Definition 5. *A* Muller automaton *is a pair* $(\mathcal{A}, \mathcal{T})$*, where* $\mathcal{A} = (\mathcal{Q}, \Sigma, \delta, \mathcal{Q}_{in})$ *is an automaton and* $\mathcal{T} = \langle \mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_k \rangle$ *is an* acceptance table *with* $\mathcal{Q}_i \subseteq \mathcal{Q}$ *for* $i \in \{1, 2, \ldots, k\}$*.*

The Muller automata accepts *(or* recognizes*) an input* $w \in \Sigma^\omega$ *if there is a run* ρ *of* $\mathcal{A}$ *on* w *such that* $\{q \in \mathcal{Q} \mid \exists^\omega n : \rho_n = q\} = F_i$ *for some* i*.*

The accepting condition is quite strong: for a word to be accepted, there must exist a run ρ and an entry F_i of the acceptance table, such that the set of states visited infinitely often by ρ is exactly F_i.